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On the Roots of Matrices.

By W. H. Metzler.

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Introduction.

In his memoir on Matrices (Phil. Trans. 1858) Prof. Cayley enunciated the theorem: "The determinant, having for its matrix a given matrix less the same matrix considered as a single quantity involving the matrix unity, is equal to zero." The equation implied in this theorem is known as Cayley's "identical equation." Subsequently (in the Mess. Math. Vol. XIII, p. 139), Mr. A. R. Forsyth gave a proof of this identical equation for matrices of the third order, based upon the solution of a system of linear difference equations.* Forsyth's method is applicable to matrices of any order. Considerable simplicity is gained, however, by the employment of non-scalar equations instead of the scalar equations employed by Forsyth.

I have employed this modification of Forsyth's method to prove Sylvester's law of latency and Sylvester's theorem. In addition I have by this method investigated the existence of roots of matrices for different indices and in particular the roots of nilpotent matrices.

For valuable suggestions in the working of this paper I am indebted to Dr. Henry Taber.

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*Sylvester stated (in the Johns Hopkins Univ. Circ. No. 28, 1884) that a proof of the identical equation could be obtained by the method of linear difference equations.

§1.—*Representation of a Matrix.*

In obtaining the representation of matrices by the method of difference equations, I shall make two general divisions: I, in which all latent roots are different from zero, and II, in which some latent roots are zero.

I. *Latent roots* $\neq 0$.—When the latent roots are different from zero, it will be found convenient to distinguish between two cases (a), in which all the latent roots are distinct, and (b), in which there are groups or sets of equal latent roots.

(a). *All latent roots distinct.*

1. The method readily presents itself on considering a few examples, as follows:

1).—*Matrix of order* 2.

Suppose $\phi = \begin{Bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{Bmatrix}$, where ϕ_{11}, ϕ_{12}, etc., represent the constituents in the positions indicated by their suffices.

Define integer powers of ϕ by $\phi^{n+1} = \phi \,.\, \phi^n$; and let

$$\phi^n = \begin{Bmatrix} (\phi^n)_{11} & (\phi^n)_{12} \\ (\phi^n)_{21} & (\phi^n)_{22} \end{Bmatrix}.$$

We have then

$$\begin{Bmatrix} (\phi^{n+1})_{11} & (\phi^{n+1})_{12} \\ (\phi^{n+1})_{21} & (\phi^{n+1})_{22} \end{Bmatrix} = \begin{Bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{Bmatrix} \begin{Bmatrix} (\phi^n)_{11} & (\phi^n)_{12} \\ (\phi^n)_{21} & (\phi^n)_{22} \end{Bmatrix}.$$

Expanding the right-hand member we get the equations

$$\begin{aligned} (\phi^{n+1})_{11} &= \phi_{11}(\phi^n)_{11} + \phi_{12}(\phi^n)_{21}, \\ (\phi^{n+1})_{21} &= \phi_{21}(\phi^n)_{11} + \phi_{22}(\phi^n)_{21}, \\ &\text{etc., etc.} \end{aligned}$$

The left-hand member of the first equation is $E(\phi^n)_{11}$, and of the second equation is $E(\phi^n)_{21}$, where E is the enlargement symbol of finite differences.

If we multiply the first equation by ϕ_{22} and subtract from the product the second equation multiplied by ϕ_{12}, we get

$$\{E^2 - (\phi_{11} + \phi_{22})E + \phi_{11}\phi_{22} - \phi_{12}\phi_{21}\}(\phi^n)_{11} = 0,$$

or $(E - g_1)(E - g_2)(\phi^n)_{11} = 0$, where g_1 and g_2 are the roots of the equation

$$\begin{vmatrix} \phi_{11} - x & \phi_{12} \\ \phi_{21} & \phi_{22} - x \end{vmatrix} = 0.$$

The function $\begin{vmatrix} \phi_{11} - x & \phi_{12} \\ \phi_{21} & \phi_{22} - x \end{vmatrix}$ is called the *latent function* of ϕ;* g_1 and g_2 are called the *latent roots* of ϕ.

Similarly we obtain

$$(E - g_1)(E - g_2)(\phi^n)_{12} = 0,$$

and generally

$$(E - g_1)(E - g_2)(\phi^n)_{rs} = 0, \qquad \left.\begin{matrix} r \\ s \end{matrix}\right\} = 1,\ 2.$$

The solution of this difference equation is

$$(\phi^n)_{rs} = A_{rs} g_1^n + B_{rs} g_2^n,$$

where A_{rs} and B_{rs} are constants determined by giving n successively the values 0 and 1.

$$\therefore\ (1)_{rs}\dagger = A_{rs} + B_{rs},$$

$$\phi_{rs} = A_{rs} g_1 + B_{rs} g_2;$$

$$\therefore\ A_{rs} = \frac{\begin{vmatrix} (1)_{rs} & 1 \\ \phi_{rs} & g_2 \end{vmatrix}}{\Delta},\ B_{rs} = \frac{\begin{vmatrix} 1 & (1)_{rs} \\ g_1 & \phi_{rs} \end{vmatrix}}{\Delta} \text{ and } \Delta = \begin{vmatrix} 1 & 1 \\ g_1 & g_2 \end{vmatrix}.$$

We may write instead of the above equation,

$$\phi = A_0 g_1 + B_0 g_2, \text{ where } A_0 = \frac{\begin{vmatrix} 1 & 1 \\ \phi & g_2 \end{vmatrix}}{\Delta}$$

$$\text{and } B_0 = \frac{\begin{vmatrix} 1 & 1 \\ g_1 & \phi \end{vmatrix}}{\Delta};$$

(i. e., we may substitute for the scalar difference equations non-scalar ones), since

$$\phi^n = \left(\begin{array}{cc} \dfrac{g_1^n(g_2 - \phi_{11})}{\Delta} + \dfrac{g_2^n(\phi_{11} - g_1)}{\Delta} & \dfrac{g_1^n(-\phi_{12}) + g_2^n\phi_{12}}{\Delta} \\ \dfrac{g_1^n(-\phi_{21}) + g_2^n\phi_{21}}{\Delta} & \dfrac{g_1^n(g_2 - \phi_{12}) + g_1^n(\phi_{22} - g_1)}{\Delta} \end{array} \right)$$

$$= \frac{g_1^n(g_2 - \phi)}{\Delta} + \frac{g_2^n(\phi - g_1)}{\Delta}$$

$$= A_0 g_1^n + B_0 g_2^n, \text{ which is the solution of } (E - g_1)(E - g_2)\phi^n = 0.$$

Similarly it may be shown for matrices of any order.

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* In general of $\phi = \left(\begin{array}{ccc} \phi_{11}\phi_{12} & \cdots\cdots & \phi_{1\omega} \\ \phi_{21}\phi_{22} & \cdots\cdots & \phi_{2\omega} \\ \vdots & & \vdots \\ \phi_{\omega 1} & \cdots\cdots\cdots & \phi_{\omega\omega} \end{array} \right)$, then $\begin{vmatrix} \phi_{11} - x & \phi_{12} \cdots\cdots & \phi_{1\omega} \\ \phi_{21} & \phi_{22} - x \cdot\cdot & \phi_{2\omega} \\ \vdots & \vdots & \vdots \\ \phi_{\omega 1} & \cdots\cdots\cdots\cdots & \phi_{\omega\omega} - x \end{vmatrix}$ is the latent function of ϕ.

† $(1)_{rs} = \begin{cases} 0 \text{ for } r \neq s \\ 1 \text{ `` } r = s \end{cases}.$

2).—*Matrix of order* 3.

Suppose $\phi = \begin{vmatrix} \phi_{11} & \phi_{12} & \phi_{13} \\ \phi_{21} & \phi_{22} & \phi_{23} \\ \phi_{31} & \phi_{32} & \phi_{33} \end{vmatrix}$ and $\phi^n = \begin{vmatrix} (\phi^n)_{11} & (\phi^n)_{12} & (\phi^n)_{13} \\ (\phi^n)_{21} & (\phi^n)_{22} & (\phi^n)_{23} \\ (\phi^n)_{31} & (\phi^n)_{32} & (\phi^n)_{33} \end{vmatrix}$.

Proceeding as in the previous case and forming the difference equation, we get $(E-g_1)(E-g_2)(E-g_3)\phi^n = 0$, where g_1, g_2, g_3 are the latent roots of ϕ.

The solution of this is

$$\phi^n = A_0 g_1^n + B_0 g_2^n + C_0 g_3^n,$$

$$\text{and } A_0 = \frac{\begin{vmatrix} 1 & 1 & 1 \\ \phi & g_2 & g_3 \\ \phi^2 & g_2^2 & g_3^2 \end{vmatrix}}{\Delta}, \quad B_0 = \frac{\begin{vmatrix} 1 & 1 & 1 \\ g_1 & \phi & g_3 \\ g_1^2 & \phi^2 & g_3^2 \end{vmatrix}}{\Delta}$$

$$C_0 = \frac{\begin{vmatrix} 1 & 1 & 1 \\ g_1 & g_2 & \phi \\ g_1^2 & g_2^2 & \phi^2 \end{vmatrix}}{\Delta}, \quad \Delta = \begin{vmatrix} 1 & 1 & 1 \\ g_1 & g_2 & g_3 \\ g_1^2 & g_2^2 & g_3^2 \end{vmatrix}$$

3).—*Matrix of order* ω.

Suppose ϕ has as latent roots $g_1, g_2, g_3, \ldots\ldots g_\omega$.

Forming the difference equation we get

$$(E-g_1)(E-g_2)\ldots\ldots(E-g_\omega)\phi^n = 0,$$

the solution of which is

$$\phi^n = A_0 g_1^n + B_0 g_2^n + \ldots\ldots + W_0 g_\omega^n.$$

Giving n successively the ω different values $0, 1, \ldots\ldots, \omega-1$, we get ω equations linear in A_0, B_0, etc., which are sufficient to determine A_0, B_0, etc., as follows:

$$A_0 = \frac{\begin{vmatrix} 1 & 1 & \ldots\ldots & 1 \\ \phi & g_2 & \ldots\ldots & g_\omega \\ \vdots & \vdots & \ldots\ldots & \vdots \\ \vdots & \vdots & \ldots\ldots & \vdots \\ \phi^{\omega-1} & g_2^{\omega-1} & \ldots & g_\omega^{\omega-1} \end{vmatrix}}{\Delta}, \text{ and similarly, for } B_0,\ C_0, \text{ etc.,}$$

$$\Delta = \begin{vmatrix} 1 & 1 & \ldots\ldots & 1 \\ g_1 & g_2 & \ldots\ldots & g_\omega \\ \vdots & \vdots & \ldots\ldots & \vdots \\ \vdots & \vdots & \ldots\ldots & \vdots \\ g_1^{\omega-1} & g_2^{\omega-1} & \ldots & g_\omega^{\omega-1} \end{vmatrix}.$$

2. *Powers of ϕ.*—The matrix all of whose constituents are zero except those along the principal diagonal, which are units, is called the *matrix unity* and is generally denoted by 1.

If from the array of constituents representing a matrix ϕ we form another matrix Φ by replacing each constituent of the first array by the logarithmic differential derivative with respect to that constituent of the determinant of the array, then the product of ϕ and the transverse of Φ in either order is the matrix unity.

The transverse of Φ, written $\breve{\Phi}$, is denoted by ϕ^{-1} and is termed the *reciprocal* of ϕ.

From this definition of the reciprocal, which is as given by Sylvester (Am. Jour., Vol. VI), we see that a matrix has no reciprocal when any of its latent roots are zero.

We may observe that the expressions for A_0, B_0, etc., as given in the preceding examples, are functions of ϕ containing the $(\omega-1)^{\text{th}}$ and lower powers; and consequently we have a formula for expressing all other powers of ϕ in terms of these.

3. A_0, B_0, *etc., are idempotent and mutually nilfactorial.*—The formula for ϕ^n gives

$$\phi^\omega = A_0 g_1^\omega + B_0 g_2^\omega + \dots\dots + W_0 g_\omega^\omega,$$

$$\text{or } \phi^\omega = \frac{(\phi-g_2)(\phi-g_3)\dots\dots(\phi-g_\omega)}{(g_1-g_2)(g_1-g_3)\dots\dots(g_1-g_\omega)}g_1^\omega + \frac{(\phi-g_1)(\phi-g_3)\dots\dots(\phi-g_\omega)}{(g_2-g_1)(g_2-g_3)\dots\dots(g_2-g_\omega)}g_2^\omega + \dots\dots$$

$$+ \frac{(\phi-g_1)(\phi-g_2)\dots\dots(\phi-g_{\omega-1})}{(g_\omega-g_1)(g_\omega-g_2)\dots\dots(g_\omega-g_{\omega-1})}g_\omega^\omega,$$

$$\text{since } A_0 = \frac{(\phi-g_2)(\phi-g_3)\dots\dots(\phi-g_\omega)}{(g_1-g_2)(g_1-g_3)\dots\dots(g_1-g_\omega)},$$

$$B_0 = \frac{(\phi-g_1)(\phi-g_3)\dots\dots(\phi-g_\omega)}{(g_2-g_1)(g_2-g_3)\dots\dots(g_2-g_\omega)}, \text{ etc.}$$

Writing a scalar symbol x for ϕ in the above formula we have

$$x^\omega = \frac{(x-g_2)(x-g_3)\dots\dots(x-g_\omega)}{(g_1-g_2)(g_1-g_3)\dots\dots(g_1-g_\omega)}g_1^\omega + \frac{(x-g_1)(x-g_3)\dots\dots(x-g_\omega)}{(g_2-g_1)(g_2-g_3)\dots\dots(g_2-g_\omega)}g_2^\omega + \text{etc.}\dots,$$

an equation in x of the degree ω, whose roots are $g_1, g_2 \dots\dots g_\omega$, as is evident on substituting g_1 for x, g_2 for x, etc., in the equation. The equation may then be written as follows:

$$(x-g_1)(x-g_2)\ldots\ldots(x-g_\omega)=0, \text{ or replacing } x \text{ by } \phi \text{ we have}$$
$$(\phi-g_1)(\phi-g_2)\ldots\ldots(\phi-g_\omega)=0.$$

Again the formula for ϕ^n gives

$$1=A_0+B_0+\ldots\ldots+W_0;$$
$$\therefore\quad A_0=A_0^2,$$
$$B_0=B_0^2, \text{ etc.,}$$

since A_0B_0, A_0C_0, etc., contain all the factors of the above equation and therefore vanish.

This proves that the letters A_0, B_0, etc., are idempotent and mutually nilfactorial.

4. *Rational function of* ϕ.—Having obtained expressions for powers of ϕ, we come naturally to the consideration of a rational integral function of ϕ of an order not less than ω, which may be written as follows:

$$\sum_0^n{}^{\lambda} a_\lambda\phi^\lambda=\sum_0^n{}^{\lambda} a_\lambda[A_0g_1^\lambda+B_0g_2^\lambda+\ldots\ldots+W_0g_\omega^\lambda],$$

or
$$F\phi=A_0Fg_1+B_0Fg_2+\ldots\ldots+W_0Fg_\omega.$$

We saw that a matrix ϕ had a reciprocal provided none of its latent roots were zero, and, since a rational integral function of a matrix is a matrix, the rational integral function $F\phi$ will have a reciprocal $(F\phi)^{-1}$ provided none of its latent roots are zero.

The reciprocal of $F\phi$ would be written

$$(F\phi)^{-1}=A_0(Fg_1)^{-1}+B_0(Fg_2)^{-1}+\ldots\ldots+W_0(Fg_\omega)^{-1};$$

because
$$F\phi\,.\,(F\phi)^{-1}=A_0+B_0+\ldots\ldots+W_0$$
$$=1.$$

We may write then a rational function of ϕ as follows:

$$f\phi=\frac{F_1\phi}{F_2\phi}=\frac{A_0F_1g_1+B_0F_1g_2+\ldots\ldots+W_0F_1g_\omega}{A_0F_2g_1+B_0F_2g_2+\ldots\ldots+W_0F_2g_\omega},$$

where $F_1\phi$ and $F_2\phi$ are rational integral functions of ϕ and where none of the latent roots of $F_2\phi$ are zero. This function may be written

$$f\phi = A_0\frac{F_1g_1}{F_2g_1} + B_0\frac{F_1g_2}{F_2g_2} + \dots + W_0\frac{F_1g_\omega}{F_2g_\omega},$$
$$\text{or } f\phi = A_0fg_1 + B_0fg_2 + \dots + W_0fg_\omega.$$

5. *Identical equation.*—The formula for ϕ^n gives

$$\phi^\omega = A_0g_1^\omega + B_0g_2^\omega + \dots + W_0g_\omega^\omega,$$

and, as we have already observed, A_0, B_0, etc., are functions of the first $(\omega - 1)$ powers of ϕ and unity, and consequently the above is an equation between the first ω powers of ϕ and unity. No other equation of this or lower order will be satisfied by ϕ, since A_0, B_0, etc., are linearly independent and therefore this is Cayley's "identical equation."

The A_0, B_0, etc., may readily be shown to be linearly independent; for if they are not, suppose the relation

$$a_0A_0 + b_0B_0 + \dots + w_0W_0 = 0, \text{ where } a_0, b_0 \dots w_0$$

are scalar constants. Multiplying this by A_0 we get

$$a_0A_0 = 0,$$

since A_0 is idempotent and nilfactorial with respect to all the other letters B_0, C_0, etc., as shown in Art. 3; therefore $a_0 = 0$.

Similarly we may show that $b_0 = c_0 = \dots = w_0 = 0$ and there is therefore no linear relation between these letters.

Again we have

$$\begin{aligned}
\phi &= A_0g_1 + Bg_2 + \dots + W_0g_\omega,\\
\phi - g_1 &= B_0(g_2 - g_1) + C_0(g_3 - g_1) + \dots + W_0(g_\omega - g_1),\\
\phi - g_2 &= A_0(g_1 - g_2) + C_0(g_3 - g_2) + \dots + W_0(g_\omega - g_2),\\
&\vdots\\
\phi - g_\omega &= A_0(g_1 - g_\omega) + B_0(g_2 - g_\omega) + \dots + V_0(g_{\omega-1} - g_\omega).
\end{aligned}$$

Then since A_0, B_0, etc., are mutually nilfactorial we have

$$(\phi - g_1)(\phi - g_2) \dots (\phi - g_\omega) = 0,$$

which is the identical equation in product form. Here again it is obvious that ϕ satisfies no other equation of this or lower order, since the letters A_0, B_0, etc., are linearly independent.

Hereafter when I use the term "the letters" without further specification, I shall mean the A's, B's, etc.

6. *Sylvester's Formula.*—The rational function of Art. 4, when written in the form

$$f\phi = A_0 fg_1 + B_0 fg_2 + \ldots\ldots + W_0 fg_\omega ,$$

is obviously Sylvester's formula for the particular case of a rational function, since the expression for A_0 is $\frac{(\phi - g_2)(\phi - g_3)\ldots\ldots(\phi - g_\omega)}{(g_1 - g_2)(g_1 - g_3)\ldots\ldots(g_1 - g_\omega)}$ and similarly for the other letters.

We have then, thus far, a means of reducing a rational function of ϕ to a rational integral function containing only the first ω powers of ϕ, beginning with $\phi^0 = 1$.

(b). *Sets of equal roots.*

7. Let us for convenience take the three examples that we considered in (a), where now we suppose some of the roots to become equal.

1). *Matrix of order* 2.—Suppose in this case $g_1 = g_2$, then proceeding as in Art. 1, we find for the difference equation

$$(E - g_1)^2 \phi^n = 0 ,$$

the solution of which is

$$\phi^n = (A_0 + nA_1)\, g_1^n .$$

If $n = 0$, then $1 = A_0$,
" $n = 1$, " $\phi = (A_0 + A_1)\, g_1$;

$$\therefore \quad A_1 = \frac{\phi - g_1}{\Delta}, \text{ and } \Delta = g_1 .$$

2). *Matrix of order* 3.—In example 2) of Art. 1, put $g_3 = g_1$, then the difference equation becomes

$$(E - g_1)^2 (E - g_2)\, \phi^n = 0 .$$

The solution of this is

$$\phi^n = (A_0 + nA_1)\, g_1^n + B_0 g_2^n .$$

The expressions for the A's and B_0 may be found as before.

3). *Matrix of order* ω.—Suppose we have a matrix of order ω having as latent roots $g_1, g_2 \ldots\ldots g_r, g_s$ of multiplicities $p_1, p_2 \ldots\ldots p_r, p_s$, respectively. The difference equation then becomes

$$(E - g_1)^{p_1} (E - g_2)^{p_2} \ldots\ldots (E - g_s)^{p_s} \phi^n = 0 ;$$

$$\begin{aligned} \therefore \quad \phi^n &= (A_0 + nA_1 + n^2 A_2 + \ldots\ldots + n^{p_1 - 1} A_{p_1 - 1})\, g_1^n + (B_0 + nB_1 + \ldots\ldots \\ &\quad + n^{p_2 - 1} B_{p_2 - 1})\, g_2^n + \ldots\ldots + (S_0 + nS_1 + \ldots\ldots + n^{p_s - 1} S_{p_s - 1})\, g_s^n \\ &= g_1^n \sum_0^{p_1 - 1} {}^{\lambda}\, n^\lambda A_\lambda + g_2^n \sum_0^{p_2 - 1} {}^{\lambda}\, n^\lambda B_\lambda + \ldots\ldots + g_s^n \sum_0^{p_s - 1} {}^{\lambda}\, n^\lambda S_\lambda . \end{aligned}$$

The expressions for the A's, B's, etc., are obtained as before by giving n, ω different values, when we get ω linear equations for their determination. We shall find that

$$\Delta = \begin{vmatrix} 1 & 0 & 0 & \cdots 1 & \cdots 1 & 0 & \cdots & 0 \\ g_1 & g_1 & g_1 & \cdots g_2 & \cdots g_s & g_s & \cdots & g_s \\ g_1^2 & 2g_1^2 & 4g_1^2 & \cdots g_2^2 & \cdots g_s^2 & 2g_s^2 & \cdots & 2^{p_s-1}g_s^2 \\ g_1^3 & 3g_1^3 & 9g_1^3 & \cdots g_2^3 & \cdots g_s^3 & 3g_s^3 & \cdots & 3^{p_s-1}g_s^3 \\ \cdot & \cdot & \cdot & \cdots\cdot & \cdots\cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots\cdot & \cdots\cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots\cdot & \cdots\cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots\cdot & \cdots\cdot & \cdot & \cdots & \cdot \\ g_1^{\omega-1} & (\omega-1)\,g_1^{\omega-1} & (\omega-1)^2 g_1^{\omega-1} & \cdots g_2^{\omega-1} & \cdots g_s^{\omega-1} & (\omega-1)g_s^{\omega-1} & \cdots & (\omega-1)^{p_s-1}g_s^{\omega-1} \end{vmatrix}.$$

The factors of Δ may be found without serious difficulty. They are as follows:

$$\Delta = P \,.\, \prod_{\alpha=1}^{\alpha=s} g_\alpha^{p'_\alpha} \,.\, \prod_{\beta=1}^{\beta=s} \prod_{\gamma=1}^{\gamma=s} (g_\beta - g_\gamma)^{p_\beta p_\gamma},$$

where $\beta \gtrless \gamma$,

$$P = (p_1 - 1)!\,(p_1 - 2)! \ldots 2!\,(p_2 - 1)! \ldots (p_s - 1)! \ldots 2!,$$

and
$$p'_\alpha = (p_\alpha - 1) + (p_\alpha - 2) + \ldots + 2 + 1$$
$$= \frac{p_\alpha(p_\alpha - 1)}{2}.$$

Let us for convenience use $\Delta\binom{\phi}{\mu}$ to denote the determinant formed from Δ by substituting for its μ^{th} column the column $\begin{vmatrix} 1 \\ \phi \\ \phi^2 \\ \vdots \\ \phi^{\omega-1} \end{vmatrix}$; also let Σp_λ denote $p_1 + p_2 + \ldots + p_\lambda$, where λ has any of the values $1, 2 \ldots r$. Then

$$\frac{\Delta\binom{\phi}{1}}{\Delta} = A_0, \quad \frac{\Delta\binom{\phi}{p_1+1}}{\Delta} = B_0, \text{ etc.}$$

We get

$$\Delta\binom{\phi}{1} = P\,.\prod_{\alpha=1}^{\alpha=s} g_\alpha^{p'_\alpha}\,.\,\frac{\prod\limits_{\beta=1}^{\beta=s}(\phi-g_\beta)^{p_\beta}}{(\phi-g_1)^{p_1}}$$
$$\times F_{p_1-1}(\phi-g_1)\,.\prod_{\gamma=2}^{\gamma=s}(g_1-g_\gamma)^{(p_1-1)(p_\gamma-1)}\,.\prod_{\delta=2}^{\delta=s}\ \prod_{\epsilon=2}^{\epsilon=s}(g_s-g_\epsilon)^{p_\delta p_\epsilon},$$

$\vdots$ . . . .
. . . .

$$\Delta\binom{\phi}{\Sigma p_\lambda+1} = P\,.\prod_{\alpha=1}^{\alpha=s} g_\alpha^{p'_\alpha}\,.\,\frac{\prod\limits_{\beta=1}^{\beta=s}(\phi-g_\beta)^{p_\beta}}{(\phi-g_{\lambda+1})^{p_{\lambda+1}}}$$
$$\times F^{(p_\lambda)}_{p_{\lambda+1}-1}(\phi-g_{\lambda+1})\,.\prod_{\gamma=1}^{\gamma=s}(g_{\lambda+1}-g_\gamma)^{(p_{\lambda+1}-1)(p_\gamma-1)}\,.\prod_{\delta=1}^{\delta=s}\ \prod_{\epsilon=1}^{\epsilon=s}(g_\delta-g_\epsilon)^{p_\delta p_\epsilon},$$

where $(\lambda+1) \gtrless \delta \gtrless \varepsilon \gtrless (\lambda+1) \gtrless \gamma$, and where λ has any of the series of values mentioned before.

The factor $F^{(p_\lambda)}_{p_{\lambda+1}-1}(\phi-g_{\lambda+1})$ is a function of $(\phi-g_{\lambda+1})$ of the order $(p_{\lambda+1}-1)$, which does not contain as a factor ϕ less either of its latent roots, and is in general different for each of the letters. This formula gives the expression for the numerators of all letters with the subscript zero, and for the numerators of the letters with subscripts other than zero we have

$$\Delta\binom{\phi}{\Sigma p_\lambda+\nu+1} = P^{(p_\lambda+\nu)}\,.\,\frac{\prod\limits_{\alpha=1}^{\alpha=s} g_\alpha^{p'_\alpha}}{g_{\lambda+1}^{p_{\lambda+1}-1}}\,.\,\frac{\prod\limits_{\beta=1}^{\beta=s}(\phi-g_\beta)^{p_\beta}}{(\phi-g_{\lambda+1})^{p_{\lambda+1}-\nu}}\,.\,F^{(p_\lambda+\nu)}_{p_{\lambda+1}}(\phi-g_{\lambda+1})$$
$$\times\prod_{\gamma=1}^{\gamma=s}(g_{\lambda+1}-g_\gamma)^{(p_{\lambda+1}-1)p_\gamma}\,.\prod_{\epsilon=1}^{\epsilon=s}(g_\delta-g_\epsilon)^{p_\delta p_\epsilon},$$

where ν has any of the series of values $1,\ 2,\ \ldots\ \overline{p_{\lambda+1}-1}$, and where

$$\gamma \gtrless (\lambda+1) \gtrless \delta \gtrless \varepsilon \gtrless (\lambda+1).$$

The numerical factor remains the same for all the letters with subscript zero, but varies for the others.

Introducing now the denominator Δ on both sides we get

$$A_0 = \prod_{\beta=2}^{\beta=s}(\phi-g_\beta)^{p_\beta}\,.\,F_{p_1-1}(\phi-g_1) \div \prod_{\beta=2}^{\beta=s}(g_1-g_\beta)^{(p_1+p_\beta-1)},$$

$\vdots$ . . . .

$$A_i = P^{(i)}\,.\prod_{\beta=2}^{\beta=s}(\phi-g_\beta)^{p_\beta}\,.\,(\phi-g_1)^i\,.\,F^{(i)}_{p_1-i-1}(\phi-g_1) \div g_1^{p_1-1}\prod_{\beta=1}^{\beta=s}(g_1-g_\beta)^{p_\beta}\,.\,P;$$

and similarly for the other letters.

8. *Powers of* ϕ.—The difference equation gives an expression for ϕ^n, and (since the latent roots of ϕ are different from zero) n may have any integral value positive, negative, or zero; and here again the A's, B's, etc., are functions of ϕ containing the $(\omega - 1)^{\text{th}}$ and lower powers, so that we have a formula for expressing in terms of these all other powers of ϕ.

9. *Rational function of* ϕ.—Here as in Art. 4 we may obtain any rational integral function of ϕ, and hence any rational function of ϕ, viz. $f\phi = \frac{F_1\phi}{F_2\phi}$, provided none of the latent roots of $F_2\phi$ are zero.

It may be written as follows:

$$f\phi = \sum_{0}^{n}{}^{\mu} a_\mu \phi^\mu = \sum_{0}^{n}{}^{\mu} a_\mu \left[g_1^\mu \sum_{0}^{p_1-1}{}^{\lambda} \mu^\lambda A_\lambda + g_2^\mu \sum_{0}^{p_2-1}{}^{\lambda} \mu^\lambda B_\lambda + \dots + g_s^\mu \sum_{0}^{p_s-1}{}^{\lambda} \mu^\lambda S_\lambda \right].$$

10. *Vacuity and Nullity.*—The determinant of the array of constituents forming the matrix is called the *content* of the matrix, and is denoted by $|\phi|$. If the latent function of ϕ be written in the form

$$x^\omega - m_{\omega-1}x^{\omega-1} + m_{\omega-2}x^{\omega-2} \dots \pm m_1 x \mp m = 0,$$

then it is obvious that m is the content of ϕ; m_1 is the sum of all the principal first minors of $|\phi|$, and generally m_κ is the sum of all the principal κ^{th} minors of $|\phi|$. If $m = 0$, the matrix ϕ evidently has one latent root zero and is termed *vacuous.* If $m = 0$, and $m_1 \neq 0$, then ϕ has but one latent root zero, and is then said to be simply vacuous or to have the vacuity one. More generally, if all the m's from m to $m_{\kappa-1}$ are zero, and $m_\kappa \neq 0$, then ϕ has κ latent roots zero, and is said to have the vacuity κ. If $|\phi| \neq 0$, ϕ has no latent root zero, and is then non-vacuous.

If all the $(\kappa - 1)^{\text{th}}$ minors of the content of a matrix vanish, but not all the κ^{th} minors, the matrix is said to have a *nullity* κ. The nullity may be equal to or less than the vacuity, but never can exceed it.

11. *Identical equation.*—If $g_1, g_2, \dots g_s$ of multiplicities $p_1, p_2, \dots p_s$ respectively are the latent roots of ϕ, then the latent function of ϕ may be written

$$\begin{aligned} F(x) &= (x - g_1)^{p_1}(x - g_2)^{p_2} \dots (x - g_s)^{p_s} \\ &= a_\omega x^\omega + a_{\omega-1}x^{\omega-1} + \dots + a_1 x + a_0 = 0, \end{aligned}$$

where ω is the order of the matrix.

Then
$$\begin{aligned} F(\phi) &= a_\omega\phi^\omega + a_{\omega-1}\phi^{\omega-1} + \ldots\ldots + a_1\phi + a_0 \\ &= A_0(a_\omega g_1^\omega + a_{\omega-1}g_1^{\omega-1} + \ldots\ldots + a_1g_1 + a_0) \\ &+ A_1(\omega a_\omega g_1^\omega + \overline{\omega - 1}\,.\,a_{\omega-1}g_1^{\omega-1} + \ldots\ldots + a_1g_1) \\ &+ A_2(\omega^2 a_\omega g_1^\omega + \overline{\omega - 1}^{\,2}\,.\,a_{\omega-1}g_1^{\omega-1} + \ldots\ldots + a_1g_1) + \text{etc.} \\ &+ B_0(a_\omega g_2^\omega + a_{\omega-1}g_2^{\omega-1} + \ldots\ldots + a_1g_2 + a_0) \\ &+ B_1(\omega a_\omega g_2^\omega + \overline{\omega - 1}\,.\,a_{\omega-1}g_2^{\omega-1} + \ldots\ldots + a_1g_2) \\ &+ \text{etc., etc.} \\ &= A_0Fg_1 + A_1\Big(g_1\frac{d}{dg_1}\Big)Fg_1 + \ldots\ldots + A_i\Big(g_1\frac{d}{dg_1}\Big)^iFg_1 + \ldots\ldots \\ &+ B_0Fg_2 + \ldots\ldots\ldots\ldots\ldots\ldots + B_j\Big(g_2\frac{d}{dg_2}\Big)^jFg_2 + \ldots\ldots \\ &+ \text{etc., etc.} \end{aligned}$$

But
$$\begin{aligned} &Fg_1 = F'g_1 = F''g_1 = \ldots\ldots = F^{(p_1-1)}g_1 = 0, \\ &Fg_2 = F'g_2 = \ldots\ldots\ldots\ldots = F^{(p_2-1)}g_2 = 0, \\ &\text{etc., etc.} \end{aligned}$$
$$\therefore\quad F(\phi) = (\phi - g_1)^{p_1}(\phi - g_2)^{p_2}\ldots\ldots(\phi - g_s)^{p_s} = 0.$$

In general ϕ does not satisfy an equation of lower order than the one above, in which case it is the identical equation.

When ϕ does satisfy an equation of lower order the identical equation is said to *degrade*. It is evident that ϕ satisfies but one equation of lowest order.

We have yet to prove, however, that ϕ less either of its latent roots must be contained among the factors of this equation of lowest order. We proceed in the first place to prove certain properties of the A's, B's, etc., which we require.

Having the above equation and knowing the factors of the A's, B's, etc., we see that the letters of one set are nilfactorial with respect to the letters of any other set, and that all the letters except those with subscript zero are nilpotent.

We have
$$1 = A_0 + B_0 + \ldots\ldots + S_0\,;$$
$$\therefore\quad A_0 = A_0^2,\ B_0 = B_0^2,\ \text{etc. (multiplying by } A_0,\ B_0, \text{ etc.)}\,;$$
also
$$\begin{aligned} &A_1 = A_0A_1 = A_1A_0,\ A_2 = A_0A_2 = A_2A_0,\ \text{etc.,} \\ &B_1 = B_0B_1 = B_1B_0,\ \text{etc.,} \\ &\text{etc., etc.} \end{aligned}$$

Therefore the letters with the subscript zero are idempotent and also idemfactorial with respect to all the letters of the same set.

Some of the letters with subscripts other than zero may vanish. If $A_\kappa = 0$, then all the A's with higher subscript vanish, for A_κ contains the factors $(\phi - g_2)^{p_2}(\phi - g_3)^{p_3} \dots (\phi - g_s)^{p_s}(\phi - g_1)^\kappa$ besides a homogeneous rational integral function of ϕ and its latent roots, which is non-vacuous as mentioned in Art. 7, and therefore does not affect the vanishing of A_κ; consequently $(\phi - g_2)^{p_2} \dots (\phi - g_s)^{p_s}(\phi - g_1)^\kappa = 0$. But this is a factor of all the A's with higher subscripts, they then also vanish.

The A's, B's, etc., are linearly independent. To prove this it is sufficient to show that the A's are linearly independent, for assuming a linear relation between all the letters we have but to multiply it by A_0 to get rid of all the other letters, leaving a linear relation between the A's.

Suppose $A_{p_1 - a_1 - 1}$, $B_{p_2 - a_2 - 1} \dots S_{p_s - a_s - 1}$ are the letters with the highest suffices which do not vanish, and suppose the relation

$$a_0 A_0 + a_1 A_1 + a_2 A_2 + \dots + a_{p_1 - a_1 - 1} A_{p_1 - a_1 - 1} = 0.$$

Multiply by $(\phi - g_1)^{p_1 - a_1 - 1}$;

$$\therefore \quad a_0 A_0 (\phi - g_1)^{p_1 - a_1 - 1} = 0, \text{ but } A_0(\phi - g_1)^{p_1 - a_1 - 1} \neq 0;$$

$$\therefore \quad a_0 = 0.$$

Similarly all the other coefficients may be shown to be zero and therefore no linear relation exists between the A's or any of the letters.

Now suppose

$$Fg = ag^\kappa + bg^{\kappa - 1} + \dots lg + m$$

to be a rational integral function of g, of order $\varkappa$ equal to the order of the lowest equation which ϕ satisfies. Then

$$\begin{aligned}
F\phi &= a\phi^\kappa + b\phi^{\kappa-1} + \dots + l\phi + m \\
&= A_0(ag_1^\kappa + bg_1^{\kappa-1} + \dots + lg_1 + m) \\
&\quad + A_1(\varkappa a g_1^\kappa + \overline{\varkappa - 1}\, b g_1^{\kappa-1} + \dots + lg_1) \\
&\quad + A_2(\varkappa^2 a g_1^\kappa + \overline{\varkappa - 1}^2 b g_1^{\kappa-1} + \dots + lg_1) + \text{etc.} \\
&\quad + B_0(ag_2^\kappa + bg_2^{\kappa-1} + \dots + lg_2 + m) \\
&\quad + B_1(\varkappa a g_2^\kappa + \overline{\varkappa - 1}\, b g_2^{\kappa-1} + \dots + lg_2) + \text{etc., etc.} \\
&= A_0 F g_1 + g_1 A_1 F' g_1 + g_1 A_2(F' g_1 + g_1 F'' g_1) + g_1 A_3(F' g_1 + 3 g_1 F'' g_1 + g_1^2 F''' g_1) \\
&\quad + g_1 A_4(F' g_1 + 7 g_1 F'' g_1 + 6 g_1^2 F''' g_1 + g_1^3 F'''' g_1) \\
&\quad + g_1 A_5(F' g_1 + 15 g_1 F'' g_1 + 25 g_1^2 F''' g_1 + 10 g_1^3 F'''' g_1 + g_1^4 F''''' g_1) \\
&\quad + \text{etc.} \\
&\quad + B_0 F g_2 + g_2 B_1 F' g_2 + g_2 B_2 (F' g_2 + g_2 F'' g_2) + \text{etc.} \\
&\quad \text{etc., etc.}
\end{aligned}$$

$$= A_0Fg_1 + g_1 + g_1A_1F'g_1 + \ldots\ldots + A_i\left(g_1\frac{d}{dg_1}\right)^i Fg_1 + \ldots\ldots$$
$$+ B_0Fg_2 + g_2B_1F'g_2 + \ldots\ldots + B_j\left(g_2\frac{d}{dg_2}\right)^j Fg_2 + \ldots\ldots$$
$$+ \text{ etc., etc.}$$

Since the A's, B's, etc., are linearly independent, the necessary and sufficient condition that $F\phi = 0$ is

$$\begin{array}{l} Fg_1 = 0,\ F'g_1 = 0, \ldots\ldots F^{(p_1-\alpha_1-1)}g_1 = 0; \\ Fg_2 = 0,\ F'g_2 = 0, \ldots\ldots F^{(p_2-\alpha_2-1)}g_2 = 0; \\ \vdots\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ Fg_s = 0,\ F'g_s = 0\ \ldots\ldots F^{(p_s-\alpha_s-1)}g_s = 0. \end{array}$$

These results show that the roots of $Fg = 0$ are $g_1, g_2 \ldots\ldots g_s$, of multiplicities $\overline{p_1 - \alpha_1}$, $\overline{p_2 - \alpha_2}$, $\ldots\ldots \overline{p_s - \alpha_s}$ respectively, and consequently

$$F\phi \equiv (\phi - g_1)^{p_1-\alpha_1}(\phi - g_2)^{p_2-\alpha_2} \ldots\ldots (\phi - g_s)^{p_s-\alpha_s}, \text{ (to a scalar factor),}$$

where ϕ less each of its latent roots occurs as a factor.

II. *Some latent roots zero.*—We now come to the case where some of the latent roots of ϕ are zero.

12. As before, let us consider a few examples and observe the form of the difference equation and its solution.

1). *Matrix of order* 2.

Latent roots g_1, 0.

We obtain the difference equation

$$E(E - g_1)\phi^n = 0, \text{ which gives as solution}$$
$$\phi^n = A_0g_1^n, \text{ for } n \overset{=}{>} 1;$$
$$\therefore\ \phi^n = \phi g_1^{n-1}$$

This is the same value of ϕ^n as would have been obtained by putting $g_2 = 0$ in the expression for ϕ^n, in example 1), of Art. 1.

2). *Matrix of order* 3.

Latent roots g_1, 0, 0.

The difference equation is

$$E^2(E - g_1)\phi^n = 0;$$
$$\therefore\ \phi^n = A_0g_1^n, \text{ for } n \overset{=}{>} 2;$$
$$\therefore\ \phi^n = \phi^2 g_1^{n-2}.$$

This again is the same expression for ϕ^n as we should have obtained by putting $g_2 = g_3 = 0$, in example 2) of Art. 1.

3). *Matrix of order* ω.

Let the latent roots be g_1, and 0 of multiplicity $\overline{\omega - 1}$.

The difference equation is

$$E^{\omega-1}(E - g_1)\phi^n = 0 ;$$

$$\therefore \quad \phi^n = A_0 g_1^n, \text{ for } n \geqq \overline{\omega - 1} ;$$

$$\therefore \quad \phi^n = \phi^{\omega-1} g_1^{n-\omega+1}.$$

This also is the same expression as we should have obtained by putting $g_2 = g_3 = \ldots . = g_\omega = 0$, in example 3), of Art. 1.

4). *Matrix of order* ω.

Take the most general case, where the latent roots are $g_1, g_2 \ldots . g_r$, 0, of multiplicities $p_1, p_2, \ldots . p_s$, respectively, and where $p_1 + p_2 + \ldots . + p_s = \omega$.

The difference equation is found to be

$$(E - g_1)^{p_1}(E - g_2)^{p_2} \ldots . (E - g_r)^{p_r} E^{p_s} \phi^n = 0 .$$

For $n \geqq p_s$ the solution of this is

$$\phi^n = g_1^n \sum_0^{p_1-1} {}^{\lambda} n^\lambda A_\lambda + g_2^n \sum_0^{p_2-1} {}^{\lambda} n^\lambda B_\lambda + \ldots . + g_r^n \sum_0^{p_r-1} {}^{\lambda} n^\lambda R_\lambda .$$

It is obvious that it makes no difference what values we give to n, in the solution of the difference equation, to obtain expressions for the A's, B's, etc., as long as we take any ω different values which n may have; so that in this case we get for Δ the following:

$$\Delta = \begin{vmatrix} g_1^{p_s} & p_s g_1^{p_s} & \cdots & g_2^{p_s} & \cdots & g_r^{p_s} & \cdots & g_r^{p_s} \\ g_1^{p_s+1} & (p_s+1) g_1^{p_s+1} & \cdots & g_2^{p_s+1} & \cdots & g_r^{p_s+1} & \cdots & (p_s+1)^{p_r-1} g_r^{p_s+1} \\ \vdots & \vdots & \cdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ \vdots & \vdots & \cdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ g_1^{\omega-1} & (\omega-1) g_1^{\omega-1} & \cdots & g_2^{\omega-1} & \cdots & g_r^{\omega-1} & \cdots & (\omega-1)^{p_r-1} g_r^{\omega-1} \end{vmatrix} .$$

If we examine the equations from which the A's, B's, etc., are determined, we shall easily see that the A's, B's, etc., of this case are what the A's, B's, etc., of example 3), Art. 7, become, when g_s is put equal to zero; and consequently the expression for ϕ^n found here is what that found in example 3), Art. 7, reduces to when $g_s = 0$.

For $n < p_s$, the solution of the difference equation is

$$\phi^n = g_1^n \sum_0^{p_1 - 1}{}_\lambda\, n^\lambda A_\lambda + \ldots\ldots + g_r^n \sum_0^{p_r - 1}{}_\lambda\, n^\lambda R_\lambda + N_n,$$

where N_n is some expression resulting from the solution of $E^{p_s}\phi^n = 0$, and may be determined in the same way as the expressions for the A's, B's, etc. When $n \geqq p_s$, $N_n = 0$.

If the expression for N_n be determined, as just mentioned, it will be found to be what the term $g_s^n \sum_0^{p_s - 1}{}_\lambda\, n^\lambda S_\lambda$ reduces to when $g_s = 0$. This term does not vanish when g_s becomes zero, as might appear.

We have $\qquad g_s^n S_0 = 0$, when $n \geqq 1$ and $g_s = 0$;
and $\qquad g_s^n S_\lambda = 0$, " $\;n \geqq p_s$ " $\;g_s = 0$;
as may easily be seen from their factors given in Art. 7.

That N_n is what this term reduces to when $g_s = 0$, is also apparent from the fact that the solution of

$$(E - g_1)^{p_1}(E - g_2)^{p_2} \ldots\ldots (E - g_r)^{p_r} E^{p_s}\phi^n = 0$$

is the same as what would result from putting $g_s = 0$ in the solution of

$$(E - g_1)^{p_1}(E - g_2)^{p_2} \ldots\ldots (E - g_r)^{p_r}(E - g_s)^{p_s}\phi^n = 0.$$

We arrive at the conclusion, therefore, that the formula for ϕ^n when none of its latent roots are zero, still applies when some of them become zero.

13. *Expression for unity.*—In the cases where none of the latent roots were zero we could put $n = 0$ in the solution of the difference equation and obtain an expression for unity, but when ϕ is vacuous we cannot do so. In the general case when none of the latent roots are zero we have

$$1 = A_0 + B_0 + C_0 + \ldots\ldots + S_0,$$

which is an identity; that is, if developed according to powers of ϕ, the coefficient of each power of ϕ (which is a function of the latent roots) would be identically zero. This property of the coefficients still subsists however small any of the latent roots become, and consequently is still true when one of them becomes zero. Suppose $g_s = 0$, and denote by A_0', $B_0' \ldots . S_0'$ what A_0, $B_0 \ldots . S_0$ become. We have then

$$1 = A_0' + B_0' + \ldots . + S_0',$$

which is an expression for unity when some of the latent roots are zero.

14. *Powers of* ϕ.—The difference equation gives us an expression for ϕ^n; but ϕ, being vacuous, has no reciprocal and therefore n can have only positive values. This formula for ϕ^n gives a means of expressing the ω^{th} and higher powers of ϕ as rational integral functions of the $(\omega - 1)^{\text{st}}$ and lower powers.

15. *Rational integral function of* ϕ.—Having an expression for any positive integral power of ϕ, we can write any rational integral function of an order not less than ω as follows:

$$\sum_0^n {}^{\mu} a_\mu \phi^\mu = \sum_0^n {}^{\mu} a_\mu \Big[g_1^\mu \sum_0^{p_1 - 1} {}^{\lambda} \mu^\lambda A_\lambda + \ldots . + g_r^\mu \sum_0^{p_r - 1} {}^{\lambda} \mu^\lambda R_\lambda + N_\mu \Big].$$

16. *Identical Equation.*—We have

$$\phi^\omega = (A_0 + \omega A_1 + \text{etc.}) g_1^\omega + (B_0 + \omega B_1 + \text{etc.}) g_2^\omega + \ldots . + (R_0 + \omega R_1 + \text{etc.}) g_r^\omega;$$

and it has been observed that the A's, B's, etc., here are what those of Art. 11 reduce to when $g_s = 0$. This expression for ϕ^ω is therefore what the expression of that article reduces to when $g_s = 0$, and therefore it may be written

$$(\phi - g_1)^{p_1} (\phi - g_2)^{p_2} \ldots . (\phi - g_r)^{p_r} \phi^{p_s} = 0,$$

which is the identical equation unless ϕ satisfies an equation of lower order.

Having this equation, the factors of the various letters show that (1) the letters with subscripts other than zero are nilpotent and (2) the letters of any one set are nilfactorial with respect to the letters of any other set. From the expression for unity, as given in Art. 13, it may be shown in the same manner as in Art. 11, that the letters with subscript zero are idempotent and also idemfactorial with respect to all the other letters of the same set.

Each one of the sum of S's of which N_μ is composed contains all the factors of the identical equation, and consequently N_μ is nilpotent.

It may easily be shown that all the letters, including N_μ, are linearly independent; and we have therefore sufficient data for showing in precisely the same manner as it was shown in Art. 11, that ϕ less each of its latent roots must appear as a factor in the identical equation; and also that ϕ satisfies no other equation of the same order. The identical equation may then be written

$$(\phi - g_1)^{p_1 - a_1}(\phi - g_2)^{p_2 - a_2} \ldots . (\phi - g_r)^{p_r - a_r}\phi^{p_s - a_s} = 0 .$$

§ 2.—*Properties of the A's, B's, etc.*

17. We have already proven that:

(1). The letters with the subscript zero are idempotent and idemfactorial with respect to all the other letters of the same set;

(2). The letters of any one set are nilfactorial with respect to the letters of any other set; and

(3). The letters with subscripts other than zero are nilpotent;

and we shall now establish a relation between the letters of any set.

18. *Relation between the A's.*—All the letters of any set, with subscripts greater than unity, can be expressed as powers of the letter with subscript unity according to the following law:

$$L_1^\lambda = \lambda !\ L_\lambda .$$

This relation may be established in two different ways as follows:

First Method.

We may write

$$\begin{aligned} A_0\phi &= A_0[(\phi - g_1) + g_1] \\ &= A_0[A_0(\phi - g_1) + g_1] \\ &= A_0 g_1\Big[\frac{A_0(\phi - g_1)}{g_1} + 1\Big] \\ &= A_0 g_1[N + 1], \end{aligned}$$

where N is put for $\dfrac{A_0(\phi - g_1)}{g_1}$.

Let us denote by log $(N + 1)$ the series

$$N - \frac{N^2}{2} + \frac{N^3}{3} - \frac{N^4}{4} + \ldots . + (-1)^{n-1}\frac{N^n}{n} + \text{etc.},$$

which is finite, since the p_1^{st} and all higher powers of N vanish.

If we develop $e^{\log(N+1)} = e^{N-\frac{N^2}{2}+\frac{N^3}{3}-\text{etc.}}$ according to ascending powers of N, we shall find that the coefficient of N is unity, that the coefficients of the 2nd and of all higher powers as far as we choose to go vanish, and we know that all terms containing the p_1^{st} and higher powers vanish, because N^{p_1} contains the identical equation as a factor.

We may therefore write

$$1 + N = e^{\log(1+N)};$$

$$\therefore \quad A_0\phi = A_0 g_1 e^{\log(1+N)}$$

$$= A_0 g_1 e^{A},$$

where A is put for $\log(1 + N)$.

$$A_0\phi^n = A_0 g_1^n e^{nA};$$

$$\therefore \quad A_0\phi^n = A_0 g_1^n\left(1 + nA + \frac{n^2A^2}{2} + \frac{n^3A^3}{3\,!} + \text{etc.}\right)$$

$$= \quad g_1^n\left(A_0 + nA + \frac{n^2A^2}{2} + \frac{n^3A^3}{3\,!} + \text{etc.}\right),$$

A_0 being idemfactorial with respect to the A^λ's.

The expression already found for ϕ^n is

$$\phi^n = (A_0 + nA_1 + n^2A_2 + \text{etc.})g_1^n + (B_0 + nB_1 + \text{etc.})g_2^n + \text{etc.};$$

$$\therefore \quad A_0\phi^n = (A_0 + nA_1 + n^2A_2 + \text{etc.})g_1^n.$$

These two expressions must be identical;

$$\therefore \quad A_0 + nA_1 + \text{etc.} + n^{p_1-1}A_{p_1-1} \equiv A_0 + nA + \frac{n^2A^2}{2} + \ldots\ldots + \frac{(nA)^{p_1-1}}{(p_1-1)!}.$$

It is obvious that $A^{p_1} = 0$, since $N^{p_1} = 0$;

$$\therefore \quad (A - A_1) + n\left(\frac{A^2}{2\,!} - A_2\right) + n^2\left(\frac{A^3}{n\,!} - A_3\right) + n^3\left(\frac{A^4}{4\,!} - A_4\right) + \text{etc.}$$

$$+\, n^{p_1-2}\left(\frac{A^{p_1-1}}{(p_1-1)\,!} - A_{p_1-1}\right) \equiv 0.$$

In any case this is true for all positive integral values of n, and therefore the coefficients of the various powers of n must be identically zero. We have then

$$A \equiv A_1 \text{ and generally } A^\lambda \equiv \lambda\,!\,A_\lambda.$$

Having all the A's expressed in terms of A_1 we can find the relations between them; and in an exactly similar manner the same relation may be found to exist between B's, C's, etc.

This proof of the relation is due to Dr. Henry Taber. I have found a proof of it based on the fact that

$$(A_0 + A_1 + \text{etc.})^\lambda \equiv A_0 + \lambda A_1 + \lambda^2 A_2 + \text{etc.},$$

which is somewhat long and complicated but not without some interest. It is as follows:

Second method.

For convenience suppose we have a matrix ϕ of order ω whose latent roots are $g_1, g_2 \ldots\ldots g_s$ occurring $p_1 + 1, p_2 + 1, \ldots\ldots p_s + 1$ times respectively. Write

$\overset{2}{a}_2$ for $2^2 - 2$,

$\overset{3}{a}_3$ " $3^3 - 3 - 3(2^3 - 2)$,

$\overset{4}{a}_4$ " $4^4 - 4 - \frac{4.3}{2}\{3^4 - 3 - 3(2^4 - 2)\} - 4\{2^4 - 2\}$,

$\vdots$

$\overset{p_1}{a}_{p_1}$ " $p_1^{p_1} - p_1 - \frac{p_1!}{(p_1 - 3)!\,3!}\overset{p_1}{a}_2 - \text{etc.} \ldots\ldots - \frac{p_1!}{(p_1 - k)!\,k!}\overset{p_1}{a}_\kappa - \ldots\ldots - \frac{p_1!}{(p_1 - 1)!}\overset{p_1}{a}_{p_1 - 1}$,

where $\overset{\lambda}{a}_2 = 2^\lambda - 2$,

$\overset{\lambda}{a}_3 = 3^\lambda - 3 - 3(2^\lambda - 2)$, etc., etc.;

also write

$$T \text{ for } A_1 + A_2 + \ldots\ldots + Ap_1.$$

It is readily seen that

$\overset{2}{a}_2 = 2^2 - 2$,

$\overset{3}{a}_3 = (3^3 - 3) - 3(2^3 - 2)$,

$\overset{4}{a}_4 = (4^4 - 4) - 4(3^4 - 3) + \frac{4.3}{2}(2^4 - 2)$,

$\overset{5}{a}_5 = (5^5 - 5) - 5(4^5 - 4) + \frac{5.4}{2}(3^5 - 3) - \frac{5.4.3}{2.3}(2^5 - 2)$,

$\vdots$

$$\overset{p}{a}_p = (p^p - p) - p(\overline{p-1}^p - \overline{p-1}) + \ldots\ldots + (-1)^\kappa \frac{p!}{(p-\kappa)!\,\kappa!}(\overline{p-\kappa}^p - \overline{p-\kappa}) \ldots\ldots$$

$$\ldots\ldots + (-1)^p \frac{p(p-1)}{2}(2^p - 2),$$

where the subscript of p has been dropped for convenience, since no confusion can thereby arise.

Separating the value of $\overset{p}{a}_p$ into two parts we get

$$\overset{p}{a}_p = p^p - p(p-1)^p + \dots + (-1)^\kappa \frac{p!}{(p-\kappa)!\,k!}(p-\kappa)^p + \dots + (-1)^p \frac{p(p-1)}{2} 2^p$$

$$-p + p(p-1) + \dots - (-1)^\kappa \frac{p!}{(p-\kappa-1)!\,k!} - \dots - (-1)^p p(p-1)$$

$$= p! + p(1-1)^{p-1}$$

$$= p!.$$

We know that

$$\overset{\kappa}{a}_n = 0 \text{ for } \kappa < n,$$

$$\overset{n}{a}_n = n!,$$

$$\overset{n+1}{a}_n = \tfrac{1}{2} n(n+1)!,$$

$$\overset{n+2}{a}_n = \left\{ \frac{n(n-1)}{2^3} + \frac{n}{3!} \right\}(n+2)!,$$

$$\vdots$$

$$\overset{n+\lambda}{a}_n = \Bigg[\frac{n}{(\lambda+1)!} + \frac{n(n-1)}{2} \left\{ \frac{1}{(\lambda+2)!} + \frac{2}{2!\,\lambda!} + \text{etc.} \right\}$$

$$+ \frac{n(n-1)(n-2)}{3!} \left\{ \frac{1}{(\lambda+3)!} + \frac{3}{(\lambda-1)!\,(2!)^2} + \text{etc.} \right\}$$

$$+ \dots + \frac{n!}{(\kappa+1)!\,(n-\kappa-1)} \left\{ \frac{1}{(\lambda+k+1)!} + \frac{\kappa+1}{(2!)^\kappa\,(\lambda-\kappa+1)!} + \text{etc.} \right\}$$

$$+ \text{etc.} \Bigg] (n+\lambda)!$$

$= \text{coefficient of } x^{n+\lambda} \text{ in the expansion of } (e^x - 1)^n = \left(\frac{x}{1} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \right)^n,$
multiplied by $(n+\lambda)!$.

Having obtained the foregoing auxiliary theorems we may proceed to the more direct consideration of the relation between the letters.

Taking the various powers of T up to the p^{th} we get

1). $T = A_1 + A_2 + \dots + A_p$

$\therefore \quad T^2 + 2T = 2A_1 + 2^2 A_2 + \dots + 2^p A_p,$

$\therefore$

2). $T^2 = 2!\,A_2 + \overset{3}{a}_2 A_3 + \dots + \overset{p}{a}_2 A_p \qquad = \Sigma A_1^2 + 2\Sigma A_1 A_2,$

$$3).\quad T^3 = 3\,!\,A_3 + \overset{4}{a}_3 A_4 + \ldots\ldots + \overset{p}{a}_3 A_p \qquad = \Sigma A_1^3 + 3\Sigma A_1^2 A_2 + 6\Sigma A_1 A_2 A_3,$$

$$\vdots$$

$$n).\quad T^n = n\,!\,A_n + \overset{n+1}{a}_n A_{n+1} + \ldots\ldots + \overset{p}{a}_n A_p \qquad = \Sigma A_1^n + n\Sigma A_1^{n-1} A_2 + \text{etc.},$$

$$\vdots$$

$$p).\quad T^p = p\,!\,A_p \qquad = A_1^p.$$

From these equations we easily get, on multiplying by proper powers of the various A's,

$$A_1^p = (p - \varkappa) A_1^\kappa A_{p-\kappa} \qquad = \varkappa\,!\,(p - \varkappa)\,!\,A_{p-\kappa} A_\kappa,$$

$$A_1^{p-\lambda\mu} A_\lambda^\mu = (p - \mu\lambda)\,!\,A_{p-\lambda\mu} A_\lambda^\mu,$$

etc.

$$A_1^p = (\varkappa_1\,!)^{a_1} (\varkappa_2\,!)^{a_2} \ldots\ldots (\varkappa_m\,!)^{a_m} (p - \varkappa_1^{a_1} - \varkappa_2^{a_2} - \ldots\ldots - \varkappa_m^{a_m})\,!\,A_{\kappa_1}^{a_1} \ldots\ldots A_{p - \kappa_1^{a_1} \ldots\ldots \kappa_m^{a_m}}.$$

In this way we get all relations between expressions of weight p.

From equation $p - 1$) we get

$$A_1^{p-1} + (p-1) A_1^{p-2} A_2 = (p-1)\,!\,A_{p-1} + \tfrac{1}{2}(p-1)p\,!\,A_p.$$

$$\text{But } A_1^{p-2} A_2 = \tfrac{1}{2} p\,!\,A_p;$$

$$\therefore \quad A_1^{p-1} = (p-1)\,!\,A_{p-1}.$$

Similarly, as in case of weight p, all relations between expressions of weight $(p - 1)$ can be found.

$$A_1^{p-\lambda} A_{\lambda-1} = (p - \lambda)\,!\,A_{p-\lambda} A_{\lambda-1},$$

etc. etc.

$$A_1^{p-1} = (\varkappa_1\,!)^{a_1} (\varkappa_2\,!)^{a_2} \ldots\ldots (\varkappa_m\,!)^{a_m} (p - \varkappa_1^{a_1} - \varkappa_2^{a_2} \ldots\ldots \varkappa_m^{a_m} - 1)\,!\,A_{\kappa_1}^{a_1} \ldots\ldots A_{p - \kappa_1^{a_1} \ldots\ldots \kappa_m^{a_m} - 1}.$$

From equation $p - 2$) we get

$$A_1^{p-2} + (p-2) A^{p-3} (A_2 + A_3) + \frac{(p-2)(p-3)}{2} A_1^{p-4} A_2^2$$

$$= (p-2)\,!\,A_{p-2} + \tfrac{1}{2}(p-2)(p-1)\,!\,A_{p-1} + \left\{ \frac{p-2}{3\,!} + \frac{(p-2)(p-3)}{2^3} \right\} p\,!\,A_p;$$

$$\therefore \quad A_1^{p-2} = (p-2)\,!\;A_{p-2}.$$

Again, we could proceed as before and obtain all relations between expressions of weight $(p - 2)$.

Suppose we have performed all the operations and found all the relations between expressions of the same weight to $A_1^{n+1} = (n + 1)\,!\,A_{n+1}$.

Then let us consider equation n).—

$$A_1^n + nA_1^{n-1}(A_2 + A_3 + \text{etc.}) + \frac{n(n-1)}{2}A_1^{n-2}(A_2 + A_3 + \text{etc.})^2 + \text{etc.}$$
$$= n!\,A_n + \tfrac{1}{2}n(n+1)!\,A_{n+1} + \ldots. + \overset{n+\lambda}{a_n}A_{n+\lambda} + \ldots. + \overset{p}{a_n}A_p.$$

I wish now to show that the terms of weight $(n+\lambda)$ on the one side of the equation cancel those of the same weight on the other side, where λ may have any integral value from 1 to $p-n$.

Collecting the terms of weight $n+\lambda$ on the left hand side of the equation we get

$$nA_1^{n-1}A_{\lambda+1} + \frac{n(n-1)}{2}A_1^{n-2}\Big(\overset{*}{A}{}^{2}_{\frac{\lambda+2}{2}} + 2A_2A_\lambda + 2A_3A_{\lambda-1} + \text{etc.}\Big)$$
$$+ \frac{n(n-1)(n-2)}{3!}A_1^{n-3}\Big(\overset{*}{A}{}^{3}_{\frac{\lambda+3}{3}} + 3\Sigma A_2^2A_{\lambda-1} + \text{etc.}\Big)$$
$$\ldots. + \frac{n!}{(\varkappa+1)!\,(n-\varkappa+1)!}A_1^{n-\kappa-1}\Big(\overset{*}{A}{}^{\kappa+1}_{\frac{\lambda+\kappa+1}{\kappa+1}} + (\varkappa+1)\Sigma A_2^{\kappa}A_{\lambda-\kappa+1} + \text{etc.}\Big) + \text{etc.}$$

Replacing each of the terms in this by the proper function of $A_1^{n+\lambda}$ we get

$$\Big[\frac{n}{(\lambda+1)!} + \frac{n(n-1)}{2}\Big\{\frac{1}{(\lambda+2)!} + \frac{2}{2!\,\lambda!} + \frac{2}{3!\,(\lambda-1)!} + \text{etc.}\Big\}$$
$$+ \frac{n(n-1)(n-2)}{3!}\Big\{\frac{1}{(\lambda+3)\,!} + \frac{3}{(2!)^2(\lambda-1)!} + \ldots. + \frac{6}{2!\,\varkappa!\,(\lambda-\varkappa)!} + \text{etc.}\Big\}$$
$$+ \ldots. + \frac{n!}{(\varkappa+1)!\,(n-\varkappa+1)!}\Big\{\frac{1}{\lambda+\varkappa+1)!} + \frac{\varkappa+1}{(2!)^{\kappa}(\lambda-\varkappa+1)!} + \text{etc.}\Big\}$$
$$+\text{etc.}\ldots.\Big]A_1^{n+\lambda}.$$

This is at once seen to be $\dfrac{\overset{n+\lambda}{a_n}}{(n+\lambda)!}A_1^{n+\lambda}$.

But $A_1^{n+\lambda} = (n+\lambda)!\,A_{n+\lambda}$.

Therefore the term of weight $n+\lambda$ on the left is equal to the term of weight $n+\lambda$ on the right which is $\overset{n+\lambda}{a_n}A_{n+\lambda}$;

$$\therefore\quad A_1^n = n!\,A_n,$$

where n may have any value from 2 to p, and therefore our theorem is established.

In a similar manner the same relation may be shown to exist between the letters B, C, etc.

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* These terms will not appear unless $\frac{\lambda+h}{h}$ is an integer.

19. *Relation between the* N_μ.—We have already observed that N_μ was nilpotent and nilfactorial with respect to all the other letters, and, knowing the foregoing properties of the A's, B's, etc., we may now find the relation between the N_μ of Art. 12.

$$\begin{aligned}
\phi &\equiv (A_0 + A_1 + \text{etc.})g_1 + (B_0 + B_1 + \text{etc.})g_2 + \ldots\ldots + (R_0 + R_1 + \text{etc.})g_r + N_1, \\
\phi^2 &\equiv (A_0 + 2A_1 + \text{etc.})g_1^2 + (B_0 + 2B_1 + \text{etc.})g_2^2 + \ldots\ldots + (R_0 + 2R_1 + \text{etc.})g_r^2 + N_2 \\
&\equiv \{(A_0 + A_1 + \text{etc.})g_1 + \ldots\ldots + (R_0 + R_1 + \ldots\ldots)g_r + N_1\}^2 \\
&\equiv (A_0 + 2A_1 + \text{etc.})g_1^2 + (B_0 + 2B_1 + \text{etc.})g_2^2 + \ldots\ldots + (R_0 + 2R_1 + \text{etc.})g_r^2 + N_1^2;
\end{aligned}$$

$\therefore$ $N_2 \equiv N_1^2$, and generally it will be found that $N_\mu \equiv N_1^\mu$.

I shall hereafter omit the subscript of N_1.

§3.—*Law of Latency.*

20. I shall consider the two cases, first where none of the latent roots are zero, and second where some latent roots are zero.

First case.—Suppose $A_{p_1-\alpha_1}$, $B_{p_2-\alpha_2}$ .... $S_{p_s-\alpha_s}$ are the letters with greatest subscripts which do not vanish. The rational function of Art. 9 may be written as follows:

$$\sum_{0}^{n}{}^{\mu} a_\mu \phi^\mu = \sum_{0}^{n}{}^{\mu} a_\mu \Big[g_1^\mu A_0 + g_1^\mu \sum_{1}^{p_1-\alpha_1}{}^{\lambda} \mu^\lambda A_\lambda + \ldots\ldots + g_s^\mu S_0 + g_s^\mu \sum_{1}^{p_s-\alpha_s}{}^{\lambda} \mu^\lambda S_\lambda\Big];$$

then, writing $\sum\limits_{0}^{n}{}^{\mu} a_\mu \phi^\mu = f\phi$, we have

$$\begin{aligned}
(f\phi - fg_1) = \sum_{0}^{n}{}^{\mu} a_\mu \Big[g_1^\mu \sum_{1}^{p_1-\alpha_1}{}^{\lambda} \mu^\lambda A_\lambda + (g_2^\mu - g_1^\mu)B_0 + g_2^\mu \sum_{1}^{p_2-\alpha_2}{}^{\lambda} \mu^\lambda B_\lambda + \text{etc.}& \\
+ (g_s^\mu - g_1^\mu)S_0 + g_s^\mu \sum_{1}^{p_s-\alpha_s}{}^{\lambda} \mu^\lambda S_\lambda\Big]&,
\end{aligned}$$

$$(f\phi - fg_2) = \sum_{0}^{n}{}^{\mu} a_\mu \Big[(g_1^\mu - g_2^\mu)A_0 + g_1^\mu \sum_{0}^{p_1-\alpha_1}{}^{\lambda} \mu^\lambda A_\lambda + \ldots\ldots + (g_s^\mu - g_2^\mu)S_0 + g_s^\mu \sum_{1}^{p_s-\alpha_s}{}^{\lambda} \mu^\lambda S_\lambda\Big],$$

$$\vdots$$

$$(f\phi - fg_s) = \sum_{0}^{n}{}^{\mu} a_\mu \Big[(g_1^\mu - g_s^\mu)A_0 + \ldots\ldots + g_s^\mu \sum_{1}^{p_s-\alpha_s}{}^{\lambda} \mu^\lambda S_\lambda\Big].$$

In the first equation A_0, in the second B_0, in the third C_0, etc. and in the last S_0 do not appear.

Again

$(f\phi - fg_1)^{p_1-\alpha_1}$ obviously contains none of the A's,

$(f\phi - fg_2)^{p_2-\alpha_2}$ " " " " B's,

$\vdots$

$(f\phi - fg_s)^{p_s-\alpha_s}$ " " " " S's,

$(f\phi - fg_1)^{p_1-\alpha_1}(f\phi - fg_2)^{p_2-\alpha_2}$ " " neither A's nor B's,

but contains all the other letters,

$(f\phi - fg_1)^{p_1-\alpha_1}(f\phi - fg_2)^{p_2-\alpha_2}\ldots\ldots(f\phi - fg_r)^{p_r-\alpha_r}$ contains none of the letters but S's, and

$$(f\phi - fg_1)^{p_1-\alpha_1}(f\phi - fg_2)^{p_2-\alpha_2}\ldots\ldots(f\phi - fg_s)^{p_s-\alpha_s} = 0.$$

The above expressions of the type $(f\phi - fg)$ and their products and powers are evidently the expressions of lowest orders in $f\phi$ that have the characters specified viz. as to the absence of the A's, B's, etc.

The latent roots of $f\phi$ are $fg_1, fg_2 \ldots\ldots fg_s$ of multiplicities at least $p_1 - \alpha_1$, $p_2 - \alpha_2, \ldots\ldots p_s - \alpha_s$ respectively.

If $\alpha_1 = \alpha_2 = \ldots\ldots = \alpha_s = 0$, then the latent roots of $f\phi$ are $fg_1, fg_2, \ldots\ldots fg_s$ of multiplicities $p_1, p_2 \ldots\ldots p_s$ respectively, since $\sum\limits_1^s{}_\lambda\, p_\lambda = \omega$, and $f\phi$ has ω latent roots.

Second Case.—Suppose, as in the previous case, that $A_{p_1-\alpha_1}, B_{p_2-\alpha_2} \ldots\ldots R_{p_r-\alpha_r}$ are the letters with greatest subscripts which do not vanish and that $N^{p_s-\alpha_s}$ is the greatest power of N that does not vanish. The rational integral function of Art. 15 may be written as follows:

$$F\phi = \sum_0^n{}_\mu\, a_\mu\phi^\mu = \sum_0^n{}_\mu\, a_\mu\Big[g_1^\mu A_0 + g_1^\mu\sum_1^{p_1-\alpha_1}{}_\lambda\,\mu^\lambda A_\lambda + \ldots\ldots + g_r^\mu\sum_1^{p_r-\alpha_r}{}_\lambda\,\mu^\lambda R_\lambda + N^\mu\Big].$$

Making use of the expression for unity and proceeding as before we have

$$(F\phi - Fg_1) = \sum_0^n{}_\mu\, a_\mu\Big[g_1^\mu\sum_1^{p_1-\alpha_1}{}_\lambda\,\mu^\lambda A_\lambda + (g_2^\mu - g_1^\mu)B_0 + \text{etc.}\ldots\ldots - S_0 g_1^\mu + N^\mu\Big],$$

$$(F\phi - Fg_2) = \sum_0^n{}_\mu\, a_\mu\Big[(g_1^\mu - g_2^\mu)A_0 + g_1^\mu\sum_1^{p_1-\alpha_1}{}_\lambda\,\mu^\lambda A_\lambda + g_2^\mu\sum_1^{p_2-\alpha_2}{}_\lambda\,\mu^\lambda B_\lambda + \ldots\ldots - S_0 g_2^\mu + N^\mu\Big],$$

$$\vdots$$

$$(F\phi - Fg_r) = \sum_0^n{}_\mu\, a_\mu\Big[(g_1^\mu - g_r^\mu)A_0 + \ldots\ldots + g_r^\mu\sum_1^{p_r-\alpha_r}{}_\lambda\,\mu^\lambda R_\lambda - S_0 g_r^\mu + N^\mu\Big],$$

$$(F\phi - F_0) = \sum_0^n{}_\mu\, a_\mu\Big[g_1^\mu A_0 + g_1^\mu\sum_1^{p_1-\alpha_1}{}_\lambda\,\mu^\lambda A_\lambda + \ldots\ldots + g_r^\mu\sum_1^{p_r-\alpha_r}{}_\lambda\,\mu^\lambda R_\lambda + N^\mu\Big]$$

A_0 disappears from the first equation,
B_0 " " second "
.
.
.
.
S_0 " " last "

$(F\phi - Fg_1)^{p_1-\alpha_1}$ obviously contains none of the A's,
$(F\phi - Fg_2)^{p_2-\alpha_2}$ " " " " B's,
.
.
.
.
$(F\phi - F_0)^{p_s-\alpha_s}$ " does not contain N;

and as before,

$$(F\phi - Fg_1)^{p_1-\alpha_1}(F\phi - Fg_2)^{p_2-\alpha_2}\dots\dots(F\phi - Fg_r)^{r-\alpha_r}(F\phi - F_0)^{p_s-\alpha_s} = 0\,.$$

The above expressions of the type $(F\phi - Fg)$ and their powers and products are evidently the expressions of lowest orders in $F\phi$ that have the characters specified viz., as to the absence of the A's, B's, etc.

The latent roots of $F\phi$ are Fg_1, $Fg_2\dots\dots Fg_r$, F_0, of multiplicities at least $p_1 - \alpha_1$, $p_2 - \alpha_2$, ... $p_s - \alpha_s$ respectively.

If $\alpha_1 = \alpha_2 = \dots\dots = \alpha_s = 0$ then the latent roots of $F\phi$ are $Fg_1, Fg_2, \dots\dots Fg_r, F_0$, of multiplicities $p_1, p_2, p_3, \dots\dots p_s$ respectively, since $\sum_1^s{}_\lambda\, p_\lambda = \omega$, and $F\phi$ has ω latent roots.

§4.—*Nullity of the factors of the identical equation.*

21. Let the identical equation be

$$(\phi - g_1)^{p_1-\alpha_1}(\phi - g_2)^{p_2-\alpha_2}\dots\dots(\phi - g_r)^{r-\alpha_r}\phi^{p_s-\alpha_s} = 0\,.$$

Denote all the factors of this equation except the first by ψ_1.

Then

$$(\phi - g_1)^{p_1-\alpha_1}\psi_1 = 0\,,$$

$$N_y[\phi - g_1)^{p_1-\alpha_1}\psi_1] = \omega\,,$$ where $N_y[\phi]$ denotes "the nullity of ϕ".

But $N_y[\psi_1] \leqq \omega - p_1$, since vacuity of ψ_1 is $\omega - p_1$,

and $N_y[(\phi - g_1)^{p_1-\alpha_1}] \leqq p_1$;

$\therefore\ N_y[\psi_1] = \omega - p_1$,

and $N_y[(\phi - g_1)^{p_1-\alpha_1}] = p_1$.

Similarly, $N_y[(\phi - g_\lambda)^{p_\lambda-\alpha_\lambda}] = p_\lambda$.

Let

$$\psi_1 = (\phi - g_2)^{p_2 - a_2}\psi_2.$$

We know that

$$N_y[\psi_1] = \omega - p_1,$$

or $\quad N_y[(\phi - g_2)^{p_2 - a_2}\psi_2] = \omega - p_1.$

But $\quad N_y[(\phi - g_2)^{p_2 - a_2}] \quad = p_2$ and

$N_y[\psi_2] \leqq \omega - p_1 - p_2$, since the vacuity of ψ_2 is $\omega - p_1 - p_2$;

$$\therefore \quad N_y[\psi_2] = \omega - p_1 - p_2.$$

Again

$$N_y[(\phi - g_1)^{p_1 - a_1}(\phi - g_2)^{p_2 - a_2}] \leqq p_1 + p_2,$$

$$N_y[(\phi - g_1)^{p_1 - a_1}(\phi - g_2)^{p_2 - a_2}\psi_2] = \omega;$$

$$\therefore \quad N_y[(\phi - g_1)^{p_1 - a_1}(\phi - g_2)^{p_2 - a_2}] = p_1 + p_2:$$

and generally

$$N_y[(\phi - g_1)^{p_1 - a_1}(\phi - g_2)^{p_2 - a_2} \dots (\phi - g_\lambda)^{p_\lambda - a_\lambda}] = p_1 + p_2 + \dots + p_\lambda.$$

§5.—*Roots of a Matrix.*

Under this head I shall distinguish two cases,

I. When the latent roots are all different from zero and

II. When some latent roots are zero.

I.—*Latent roots* $\neq 0$.

22. Knowing the expression for the A's, B's, etc., and the relations between them, we may write the expression for ϕ^n as follows:

$$\phi^n = A_0 g_1^n e^{nA_1} + B_0 g_2^n e^{nB_1} + \dots S_0 g_s^n e^{nS_1}.$$

Writing in this formula $n = \frac{1}{m}$ we get

$$\phi^{\frac{1}{m}} = A_0 g_1^{\frac{1}{m}} e^{\frac{A_1}{m}} + B_0 g_2^{\frac{1}{m}} e^{\frac{B_1}{m}} + \dots + S_0 g_s^{\frac{1}{m}} e^{\frac{S_1}{m}}.$$

Taking the m^{th} power of both sides we get

$$\phi = A_0 g_1 e^{A_1} + B_0 g_2 e^{B_1} + \dots + S_0 g_s e^{S_1};$$

$\therefore \quad \phi$ has an m^{th} root.

23. In the formula for $\phi^{\frac{1}{m}}$ we have s different m-valued functions, viz. $g_1^{\frac{1}{m}}, g_2^{\frac{1}{m}}, \dots g_s^{\frac{1}{m}}$, and consequently taking all possible combinations of these values we get m^s different m^{th} roots of ϕ.

24. It may be interesting to consider a few examples to show something of the character of the various functions F entering as factors in the A's, B's, etc.

1). *Matrix of order* 5.

Latent roots g_1 of multiplicity five.

$$\phi = A_0 g_1^n e^{nA_1} = (A_0 + nA_1 + n^2A_2 + n^3A_3 + n^4A_4) g_1^n,$$

$$\Delta = 4!\ 3!\ 2!\ g_1^{10},$$

$$A_0 = 1,$$

$$A_1 = -\frac{(\phi - g_1)^4}{4g_1^4} + \frac{(\phi - g_1)^3}{3g_1^3} - \frac{(\phi - g_1)^2}{2g_1^2} + \frac{(\phi - g_1)}{g_1},$$

$$A_2 = \frac{11}{24}\frac{(\phi - g_1)^4}{g_1^4} - \frac{(\phi - g_1)^3}{2g_1^3} + \frac{(\phi - g_1)^2}{2g_1^2},$$

$$A_3 = -\frac{(\phi - g_1)^4}{4g_1^4} + \frac{(\phi - g_1)^3}{6g_1^3},$$

$$A_4 = \frac{(\phi - g_1)^4}{24g_1^4}.$$

2). *Matrix of order* 6.

Latent roots all equal.

$$\phi^n = A_0 g_1^n e^{nA_1},$$

$$\Delta = 5!\ 4!\ 3!\ 2!\ g_1^{15},$$

$$A_0 = 1,$$

$$A_1 = \frac{(\phi - g_1)^5}{5g_1^5} - \frac{(\phi - g_1)^4}{4g_1^4} + \frac{(\phi - g_1)^3}{3g_1^3} - \frac{(\phi - g_1)^2}{2g_1^2} + \frac{(\phi - g_1)}{g_1},$$

$$A_2 = -\frac{5(\phi - g_1)^5}{12g_1^5} + \frac{11(\phi - g_1)^4}{24g_1^4} - \frac{(\phi - g_1)^3}{2g_1^3} + \frac{(\phi - g_1)^2}{2g_1^2},$$

$$A_3 = \frac{7(\phi - g_1)^5}{24g_1^5} - \frac{(\phi - g_1)^4}{4g_1^4} + \frac{(\phi - g_1)^3}{6g_1^3},$$

$$A_4 = -\frac{(\phi - g_1)^5}{12g_1^5} + \frac{(\phi - g_1)^4}{24g_1^4},$$

$$A_5 = \frac{(\phi - g_1)^5}{5!\ g_1^5}.$$

3). *Matrix of order* ω.

Latent roots all equal.

$$\phi = A_0 g_1^n e^{nA_1},$$

$$\Delta = (\omega - 1)!\ (\omega - 2)!\ \ldots .\ 2!\ g_1^{\frac{\omega(\omega-1)}{2}},$$

$$A_0 = 1,$$

$$A_1 = (-1)^{\omega}\frac{(\phi - g_1)^{\omega-1}}{(\omega-1)g_1^{\omega-1}} + (-1)^{\omega-1}\frac{(\phi - g_1)^{\omega-2}}{(\omega-2)g_1^{\omega-2}} + \ldots . + \frac{(\omega - g_1)}{g_1},$$

$$\vdots$$

$$A_{\omega-1} = \frac{(\phi - g_1)^{\omega-1}}{(\omega-1)!\ g_1^{\omega-1}}.$$

4). *Matrix of order* 4.

Latent roots $g_1 = g_3 = g_4,\ g_2$.

$$\phi^n = (A_0 + nA_1 + n^2 A_2)g_1^n + B_0 g_2^n,$$

$$\Delta = 2!\ g_1^3 (g_1 - g_2)^3,$$

$$A_0 = \frac{(\phi - g_2)\{(\phi - g_1)^2 + (\phi - g_1)(g_2 - g_1) + (g_2 - g_1)^2\}}{(g_1 - g_2)^3},$$

$$B_0 = \frac{(\phi - g_1)^3}{(g_2 - g_1)^3}.$$

5). *Matrix of order* ω.

Latent roots $g_1 = g_3 = g_4 = \ldots . = g_{p+1},\ g_2 = g_{p+2} = \ldots . = g_{p+q}$, where $p + q = \omega$.

$$\phi^n = A_0 g_1^n e^{nA_1} + B_0 g_2^n e^{nB_1},$$

$$\Delta = \prod_{\alpha=1}^{\alpha=p-2}(p-\alpha)!\,.\prod_{\beta=1}^{\beta=q-2}(q-\beta)!\,.\,g_1^{\frac{p(p-1)}{2}}\,g_2^{\frac{q(q-1)}{2}}\,(g_2 - g_1)^{pq},$$

$$A_0 = \frac{(\phi - g_2)^q\{k_p(\phi - g_1)^{p-1} + k_{p-1}(\phi - g_1)^{p-2}(g_2 - g_1) \ldots . + k_1(g_2 - g_1)^{p-1}\}}{(-1)^q (g_2 - g_1)^{\omega-1}},$$

$$B_0 = \frac{(\phi - g_1)^p\{h_q(\phi - g_2)^{q-1} + h_{q-1}(\phi - g_2)^{q-2}(g_1 - g_2) + \ldots . + h_1(g_1 - g_2)^{q-1}\}}{(-1)^{\omega+p-1}(g_2 - g_1)^{\omega-1}},$$

where

$$k_1 = 1,$$

$$k_2 = p,$$

$$\vdots$$

$$k_\lambda = \frac{p(p+1)\ \ldots .\ (p+\lambda-2)}{(\lambda-1)!};$$

$$h_1 = 1,$$

$$h_2 = q,$$

$$\vdots$$

$$h_\lambda = \frac{q(q+1)\ \ldots .\ (q+\lambda-2)}{(\lambda-1)!}.$$

6). *Matrix of order* 6.

Latent roots $g_1 = g_4 = g_5,\ g_2 = g_6,\ g_3$.

$$\Delta = 2!\ g_1^3 g_2 (g_2 - g_1)^6 (g_3 - g_2)^2 (g_1 - g_3)^3,$$

$$\begin{aligned} A_0 = {} & (\phi - g_2)^2 (\phi - g_3) \big[(\phi - g_1)^2 \{ 3(g_3 - g_1)^2 + 2(g_3 - g_1)(g_2 - g_1) + (g_2 - g_1)^2 \} \\ & + (\phi - g_1) \{ 2(g_3 - g_1) + (g_2 - g_1) \} (g_3 - g_1)(g_2 - g_1) + (g_3 - g_1)^2 (g_2 - g_1)^2 \big] \\ & \div (g_2 - g_1)^4 (g_3 - g_1)^3. \end{aligned}$$

25. *Negative fractional indices.*—We have seen that we may write

$$\phi^{\frac{1}{n}} = A_0 g_1^{\frac{1}{n}} e^{\frac{A_1}{n}} + B_0 g_2^{\frac{1}{n}} e^{\frac{B_1}{n}} + \dots + S_0 g_s^{\frac{1}{n}} e^{\frac{S_1}{n}}.$$

Substitute in this $-m$ for n and we have

$$\phi^{-\frac{1}{m}} = A_0 g_1^{-\frac{1}{m}} e^{-\frac{A_1}{m}} + \dots + S_0 g_s^{-\frac{1}{m}} e^{-\frac{S_1}{m}},$$

but

$$\phi^{\frac{1}{m}} = A_0 g_1^{\frac{1}{m}} e^{\frac{A_1}{m}} + \dots + S_0 g_s^{\frac{1}{m}} e^{\frac{S_1}{m}}.$$

By definition $\phi^{\frac{1}{m}} \phi^{-\frac{1}{m}} = 1$; and multiplying the corresponding sides of these two equations together we have

$$\begin{aligned} \phi^{\frac{1}{m}} \phi^{-\frac{1}{m}} &= A_0 + B_0 + \dots + S_0 \\ &= 1. \end{aligned}$$

In the formula therefore for ϕ^n when no latent root is zero, n may have any integral or fractional positive or negative value.

II. *Some latent roots zero.*—Before proceeding to the case where some but not all the latent roots are zero, I shall consider the case where all the latent roots are zero.

26.—*All latent roots zero—roots of zero.*

In what follows denote the matrix

$$\phi = \left| \begin{array}{cccccc} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right| \text{ by } 13 + 24 + 35 + 46.^*$$

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* The linear form representation of a matrix is due to Charles S. Peirce; and the notation employed here is virtually his.

The first number of each term of the sum indicating the row and the second the column in which the constituent appears, that is, ϕ is a matrix in which unity is the constituent in the first row and third column, unity is the constituent in the second row and fourth column, unity is the constituent in the third row and fifth column, and unity is the constituent in the fourth row and sixth column, all the other constituents being zero. Similarly in general.

This canonical representation of a matrix was virtually given by Buchheim (in Proc. Lond. Math. Soc. Vol. XVI), but was first explicitly given by Weyr (Comptes Rendus, Vol. C).

If g_α is an α^{uple} latent root of ϕ and if $\alpha_1,\ \alpha_1+\alpha_2, \ldots\ldots \alpha_1+\alpha_2+\alpha_3+\ldots\ldots+\alpha_\rho=\alpha$, are the nullities of the matrices $(\phi-g_\alpha)$, $(\phi-g_\alpha)^2 \ldots\ldots (\phi-g_\alpha)^\rho$, Weyr terms the numbers $(\alpha,\ \alpha_1,\ \alpha_2 \ldots\ldots \alpha_\rho)$ the *characteristics* of the latent root g_α.

I shall term two matrices of the same order *equivalent*, if they have the same latent roots with the same characteristics respectively.*

In what follows, since all the latent roots are zero, I shall speak of the characteristics of the matrix instead of the characteristics of the latent root zero.

Consider a matrix ϕ of order ω and suppose

$$N_y[\phi]=p,\ N_y[\phi^2]=p+a_1, \ldots\ldots N_y[\phi^t]=p+a_1+a_2+\ldots\ldots a_{t-1}=\omega,$$

where $p \geqq a_1 \geqq a_2 > \ldots\ldots \geqq a_{t-1}$.

If $\psi^q=\phi$, then

$$N_y[\psi^q]=p,\ N_y[\psi^{2q}]=p+a_1, \ldots\ldots N_y[\psi^{qt}]=\omega.$$

Let

$$N_y[\psi]=\alpha, \text{ then}$$

$$q\alpha \geqq p, \text{ and if } \alpha=1,\ q=p$$

$$\text{“ “ } \alpha=2,\ q \geqq \frac{p}{2}.$$

We may now establish the following results, the most of which are restrictive on q, $\frac{1}{q}$ being the index of the root.

1). It is quite obvious that if a_{t-2} is the second last increment of nullity of successive powers of ϕ, then $q \leqq a_{t-2}$.

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* Weyr defines two matrices of the same order as being “*matrices de même espèce*” if they have the same latent roots with the same characteristics; and adds that, if M and N are two equivalent matrices, one can always find a matrix Q of nullity zero such that $N=QMQ^{-1}$, which is a formula giving all equivalent matrices in terms of one of them.

2). (a). $\frac{p}{q} \leqq \alpha \leqq p - q + 1$ when $\frac{p}{q} =$ an integer.

Let $N_y[\psi] = \alpha,$

$$\begin{aligned} N_y[\psi^2] &= \alpha + \alpha_1, \\ &\vdots \\ N_y[\psi^q] &= \alpha + \alpha_1 + \ldots + \alpha_{q-1} = p. \end{aligned}$$

Then α will have its least value when $\alpha_1 + \alpha_2 + \ldots + \alpha_{q-1}$ is greatest, but this sum will be greatest when $\alpha_1 = \alpha_2 = \ldots = \alpha_{q-1} = \alpha$;

$\therefore$ α will be least when

$$\alpha + (q-1)\alpha = p,$$

$$\text{or} \quad \alpha = \frac{p}{q}; \quad \therefore \ \alpha \geqq \frac{p}{q}.$$

Again, if α is to have its greatest value, $\alpha_1 + \alpha_2 + \ldots + \alpha_{q-1}$ must have its least value, but this sum is least when $\alpha_1 = \alpha_2 = \ldots = \alpha_{q-1} = 1$;

$\therefore$ α will be greatest when

$$\alpha + q - 1 = p;$$

$$\therefore \ \alpha = p - q + 1; \quad \therefore \ \alpha \leqq p - q + 1;$$

$\therefore$ $\frac{p}{q} \leqq \alpha \leqq p - q + 1.$

(b). $\left[\frac{p}{q}\right] + 1 \leqq \alpha \leqq p - q + 1$ when $\frac{p}{q} \neq$ an integer.

Where $\left[\frac{p}{q}\right]$ denotes the greatest integer in $\frac{p}{q}$ as in previous case, α will be least when $\alpha_1 + \alpha_2 + \ldots + \alpha_{q-1}$ is greatest, that is when $\alpha_1 = \alpha_2 = \ldots = \alpha_{q-2} = \alpha$ and α_{q-1} is as great as possible;

$\therefore$ α is least when

$$\alpha(q-1) + \alpha_{q-1} = p.$$

Now α must be greater than $\left[\frac{p}{q}\right]$; for if $\alpha = \left[\frac{p}{q}\right]$ then $\alpha_{q-1} = p - q\left[\frac{p}{q}\right] + \left[\frac{p}{q}\right]$ which is obviously greater than $\left[\frac{p}{q}\right]$, that is, $\alpha_{q-1} > \alpha$, which is impossible; and therefore $\alpha > \left[\frac{p}{q}\right]$.

Let $\alpha = \left[\frac{p}{q}\right] + 1$;

$$\therefore \quad \alpha_{q-1} = p - q\left[\frac{p}{q}\right] - q + \left[\frac{p}{q}\right] + 1.$$

But $p + 1 - q\left\{\left[\frac{p}{q}\right] + 1\right\} \overset{=}{<} 0$;

$$\therefore \quad \alpha_{q-1} \overset{=}{<} \left[\frac{p}{q}\right] \text{ which is possible ;}$$

$$\therefore \quad \alpha = \left[\frac{p}{q}\right] + 1 \text{ is the lower limit.}$$

If α is to have its greatest value, $\alpha_1 + \alpha_2 + \ldots . + \alpha_{q-1}$ must have its least value;

$$\therefore \quad \alpha + q - 1 = p;$$
$$\therefore \quad \alpha \overset{=}{<} p - q + 1.$$

Therefore $\left[\frac{p}{q}\right] + 1 \overset{=}{<} \alpha \overset{=}{<} p - q + 1$.

3). If $N_y[\psi^q] = p$ and $q > \frac{p}{2}$, then $N_y[\psi^{q+1}] = p + 1$.

In this case $\alpha \overset{=}{>} 2$ and hence it is obvious that the increments of nullity must reduce to unity at or before the q^{th} power of ψ.

As an immediate consequence of this we have

$$N_y[\psi^{2q}] = p + q,\ N_y[\psi^{3q}] = p + 2q \ldots . N_y[\psi^{\kappa q}] = p + (\varkappa - 1)q;$$

and $\therefore \quad a_1 = a_2 = \ldots . = a_{t-2} = q$.

Therefore there is no q^{th} root of ϕ, q being greater than $\frac{p}{2}$, unless the nullity of successive powers of ϕ increase by equal increments of q.

4). If a_κ is the first increment of nullity that is less than $2q$, then $\psi^{(\kappa-1)q+a_\kappa+1}$ has a nullity $p + 2(\varkappa - 2) + 2a_\kappa + 1$, that is, the increment of nullity for all powers of ψ greater than the $\{(\varkappa - 1)q + a_\kappa\}^{\text{th}}$ is unity.

$$N_y[\phi^\kappa] = p + 2(\varkappa - 1)q;$$
$$\therefore \quad N_y[\psi^{\kappa q}] = p + 2(\varkappa - 1)q.$$

Let $\qquad 2x + y = a_\kappa$ and $x + y = q$;

$$\therefore \quad x = a_\kappa - q, \text{ or } y = 2q - a_\kappa.$$
$$N_y[\psi^{\kappa q + x}] = p + 2(\varkappa - 1)q + 2x,$$
$$\text{or} \quad N_y[\psi^{\kappa q + a_\kappa - q}] = p + 2(\varkappa - 1)q + 2a_\kappa - 2q;$$
$$\therefore \quad N_y[\psi^{(\kappa-1)q + a_\kappa}] = p + 2(\varkappa - 2)q + 2a_\kappa.$$

But $$N_y[\psi^{(\kappa+1)q}] = N_y[\psi^{\kappa q+x+y}] = p + 2(\varkappa-1)q + a_\kappa,$$
$$N_y[\psi^{(\kappa-1)q+a_\kappa+y}] = p + 2(\varkappa-2) + 2a_\kappa + y;$$
$$\therefore\quad N_y[\psi^{(\kappa-1)q+a_\kappa+1}] = p + 2(\varkappa-2) + 2a_\kappa + 1;$$
and therefore the increment for all powers of ψ greater than the $\{(\varkappa-1)q+a_\kappa\}^{\text{th}}$ is unity.

5). If a_κ is the first of the a's that is less than $2q$, then there is no q^{th} root of ϕ unless $a_{\kappa+1} = a_{\kappa+2} = \ldots . \, a_{t-2} = q$.

This follows as an immediate consequence of 4).

6). If a_κ is the first of the a's that is less than $2q$ and if $a_{\kappa+1} = a_{\kappa+2} = \ldots . = a_{t-2} = q$, then there always exists a q^{th} root of ϕ.

Let $N_y[\psi] = \alpha$, $N_y[\psi^2] = \alpha + 2 \ldots . N_y[\psi^q] = \alpha + 2(q-1) = p$.

Then

$$\psi = 12 + 23 + \ldots . (\overline{\varkappa-1}.q+a_\kappa-1)(\overline{\varkappa-1}.q+a_\kappa) + * + (\overline{\varkappa-1}.q+a_\kappa+1)(\overline{\varkappa-1}.q+a_\kappa+2)$$
$$+ \ldots . + \ldots . + (\omega-\alpha+1)(\omega-\alpha+2),$$

where $*$ denotes where a term, which in the natural sequence would appear, has been omitted.

7). If the nullity of successive powers of ϕ increase by equal increments of μq, $(\mu q \leqq p)$, then there is always a q^{th} root of ϕ.

For take

$$\alpha = p - \mu(q-1),$$

and give to successive powers of ψ equal increments of μ;

$$\therefore\quad N_y[\psi^q] = p.$$

And we have

$$\psi_\lambda = \lambda.\overline{\lambda+\mu} + \overline{\lambda+1}.\overline{\lambda+\mu+1} + \ldots . + \overline{\lambda+\omega-p+\mu(q-1)-1}.\overline{\lambda+\omega-p+\mu(q-1)+\mu-1},$$

where λ may take any of the values

$$1, 2, 3, \ldots . \overline{p-\mu(q-1)+1}.$$

8). If $q = \frac{p}{2} - \nu$ and $\alpha > 2\nu + 3$, then $N_y[\psi^{q-1}] = p - 1$; and therefore there is no q^{th} root of ϕ unless there are equal increments of q.

Of course $q \geqq 2$; and $\therefore p \geqq 2(\nu+2)$.

$N_y[\psi] = \alpha$, $N_y[\psi^2] = \alpha + \alpha_1$, etc.;

$$\therefore\quad \alpha + \alpha_1 + \alpha_2 + \ldots . + \alpha_{q-1} = p.$$

Suppose $\alpha_1 = \alpha_2 = \ldots\ldots = \alpha_{q-2} = 2$ and $\alpha_{q-1} = 1$;

$$\therefore \quad \alpha + (q-2)2 + 1 = p;$$

$$\therefore \quad \alpha = p - 2q + 3 = 2\nu + 3;$$

and therefore when $\alpha \gtreqless 2\nu + 3$ obviously $N_y[\psi^{q-1}] = p - 1$.

27. I propose now to show that there is always a q^{th} root of ϕ unless the law of nullity prohibits.

I shall suppose the law of nullity does not prohibit and then show that there is a q^{th} root by finding it.

Suppose $\alpha = i + j + \ldots\ldots + \varkappa + 1$

and

$$N_y[\psi^{b_0}] = b_0\alpha,\ N_y[\psi^{b_0+b_1}] = b_0\alpha + b_1(\alpha - 1) \ldots\ldots N_y[\psi^{b_0+b_1+\ldots\ldots+b_i}] = p,$$

where $q = b_0 + b_1 + b_2 + \ldots\ldots + b_i$ and

$$p = b_0\alpha + b_1(\alpha - 1) + \ldots\ldots + b_i(\alpha - i);$$

$$N_y[\psi^{q+b_{i+1}}] = p + b_{i+1}(\alpha - i - 1) \ldots\ldots N_y[\psi^{q+b_{i+1}+\ldots\ldots+b_{i+j}}] = p + a_1,$$

where

$$b_{i+1} + b_{i+2} + \ldots\ldots + b_{i+j} = q \text{ and}$$

$$b_{i+1}(\alpha - i - 1) + \ldots\ldots + b_{i+j}(\alpha - i - j) = a_1;$$

$N_y[\psi^{2q+b_{i+j+1}}] = p + a_1 + b_{i+j+1}(\alpha - i - j - 1)$, etc.,
etc.,

$N_y[\psi^{(t-1)q+b_{\alpha-\kappa}+b_{\alpha-\kappa+1}+\ldots\ldots+b_{\alpha-2}}] = p + a_1 + a_2 + \ldots\ldots + a_{t-2} + b_{\alpha-2}\cdot 2$, etc.,

$N_y[\psi^{(t-1)q+b_{\alpha-\kappa}+b_{\alpha-\kappa+1}+\ldots\ldots+b_{\alpha-1}}] = N_y[\psi^{tq}] = p + a_1 + a_2 + \ldots\ldots + b_{\alpha-1} = \omega$,

where $b_{\alpha-1}$ obviously equals a_{t-1}.

We have then for ψ the following:

$$\begin{aligned}\psi = {} & [12 + 23 + \ldots\ldots + \overline{b_0 - 1}\,.\,b_0]_{\Sigma b_0} + * + [\overline{b_0 + 1}\,.\,\overline{b_0 + 2} + \ldots\ldots \\ & \qquad + \overline{2b_0 + b_1 - 1}\,.\,\overline{2b_0 + b_1}]_{\Sigma b_1} + * + \ldots\ldots \\ & + [\overline{b_0 i + b_1(i-1) + \ldots\ldots + b_{i-1} + 1}\,.\,\overline{b_0 i + b_1(i-1) + \ldots\ldots + b_{i-1} + 2} + \ldots\ldots \\ & \qquad + \overline{b_0(i+1) + \ldots\ldots + b_i - 1}\,.\,\overline{b_0(i+1) + \ldots\ldots + b_i}]_{\Sigma b_i} \\ & + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ & + [\overline{b_0(\alpha-1) + b_1(\alpha-2) + \ldots\ldots b_{\alpha-2} + 1}\,.\,\overline{b_0(\alpha-1) + \ldots\ldots + b_{i-1} + 2} + \ldots\ldots \\ & \qquad + \overline{b_0\alpha + \ldots\ldots + b_{\alpha-1} - 1}\,.\,\overline{b_0\alpha + \ldots\ldots + b_{\alpha-1}}]_{\Sigma b_{\alpha-1}},\end{aligned}$$

where Σb_κ denotes $b_0 + b_1 + b_2 + \ldots\ldots + b_\kappa$ and where $b_0\alpha + b_1(\alpha - 1) + \ldots\ldots + b_{\alpha-1} = \omega$ the last term being therefore $(\omega - 1)\omega$.

It will be observed that ψ is divided into α sets, each of which I have enclosed in brackets with a subscript indicating the power which causes that set to vanish.

28. I shall now give all possible types of roots of nilpotent matrices of orders 3–10 inclusive.

(a).—*Matrix of order* 3.

$$N_y[\phi] = 2,\ N_y[\phi^2] = 3;$$
$$\phi = 13 = (12 + 23)^2.$$

(b).—*Matrix of order* 4.

1). $N_y[\phi] = 2,\ N_y[\phi^2] = 4;$
$$\phi = 13 + 24 = (12 + 23 + 34)^2.$$

2). $N_y[\phi] = 3,\ N_y[\phi^2] = 4;$
$$\phi = 14 = (12 + 23 + 34)^3;$$
$$\text{or } = 24 = (23 + 34)^2.$$

(c).—*Matrix of order* 5.

1). $N_y[\phi] = 2,\ N_y[\phi^2] = 4,\ N_y[\phi^3] = 5;$
$$\phi = 13 + 24 + 35 = (12 + 23 + 34 + 45)^2.$$

2). $N_y[\phi] = 3,\ N_y[\phi^2] = 5;$
$$\phi = 14 + 25 = (12 + 23 + 34 + 45)^3,$$
$$\text{or } = 24 + 35 = (23 + 34 + 45)^2.$$

3). $N_y[\phi] = 4,\ N_y[\phi^2] = 5;$
$$\phi = 15 = (12 + 23 + 34 + 45)^4 = (13 + 24 + 35)^2,$$
$$\text{or } = 25 = (23 + 34 + 45)^3,$$
$$\text{“} = 35 = (34 + 45)^2 = (12 + 34 + 45)^2.$$

Here, for the first time thus far, we have more than one type of root of the same index; and hereafter when this occurs I shall give the characteristics.

In this case we have, using *ch.* to denote “characteristics,”

$$ch.\ \phi^{\frac{1}{2}} \begin{cases} (5;\ 3,\ 1,\ 1,\ 0) \\ (5;\ 2,\ 2,\ 1,\ 0). \end{cases}$$

(d).—*Matrix of order* 6.

1). $N_y[\phi] = 2,\ N_y[\phi^2) = 4,\ N_y[\phi^3] = 6.$

Instead of indicating the nullity of successive powers of ϕ as heretofore, I shall for convenience simply write the characteristics of ϕ.

In this case we have

$ch.\ \phi\ (6;\ 2,\ 2,\ 2,\ 0);$

$\phi = 13 + 24 + 35 + 46 = (12 + 23 + 34 + 45 + 56)^2.$

2). $ch.\ \phi\ (6;\ 3,\ 3,\ 0);$

$\phi = 14 + 25 + 36 = (12 + 23 + 34 + 45 + 56)^3,$

no. sq. root vide Art. 26. 3).

3). $ch.\ \phi\ (6;\ 3,\ 2,\ 1,\ 0);$

$\phi = 24 + 35 + 46 = (23 + 34 + 45 + 56)^2.$

4). $ch.\ \phi\ (6;\ 4,\ 2,\ 0);$

$\phi = 15 + 26 = (12 + 23 + 34 + 45 + 56)^4 = (13 + 24 + 35 + 46)^2,$

or $= 25 + 36 = (23 + 34 + 45 + 56)^3,$

" $= 35 + 46 = (34 + 45 + 56)^2 = (12 + 34 + 45 + 56)^2,$

$$ch.\ \phi^{\frac{1}{2}} \begin{cases} (6;\ 3,\ 1,\ 1,\ 1,\ 0) \\ (6;\ 2,\ 2,\ 2,\ 0,) \\ (6;\ 2,\ 2,\ 1,\ 1,\ 0). \end{cases}$$

5). $ch.\ \phi\ (6;\ 5,\ 1,\ 0);$

$\phi = 16 = (12 + 23 + 34 + 45 + 56)^5,$

or $= 26 = (23 + 34 + 45 + 56)^4 = (24 + 35 + 46)^2,$

" $= 36 = (34 + 45 + 56)^3 = (12 + 34 + 45 + 56)^3,$

" $= 46 = (45 + 56)^2.$

$$ch.\ \phi^{\frac{1}{2}} \begin{cases} (6;\ 4,\ 1,\ 1,\ 0) \\ (6;\ 3,\ 2,\ 1,\ 0), \end{cases} \qquad ch.\ \phi^{\frac{1}{3}} \begin{cases} (6;\ 3,\ 1,\ 1,\ 1,\ 0) \\ (6;\ 2,\ 2,\ 1,\ 1,\ 0). \end{cases}$$

(e).—*Matrix of order* 7.

1). $ch.\ \phi\ (7;\ 2,\ 2,\ 2,\ 1,\ 0);$

$\phi = 13 + 24 + 35 + 46 + 57 = (12 + 23 + 34 + \text{etc.})^2.$

2). $ch.\ \phi\ (7;\ 3,\ 3,\ 1,\ 0);$

$\phi = 14 + 25 + 36 + 47 = (12 + 23 + 34 + \text{etc.})^3,$

no. sq. root.

3). $ch.\ \phi\ (7;\ 3,\ 2,\ 2,\ 0);$

$\phi = 24 + 35 + 46 + 57 = (23 + 34 + 45 + 56 + 67)^2.$

4). $ch.\ \phi\ (7;\ 4,\ 3,\ 0);$

$\phi = 15 + 26 + 37 = (12 + 23 + 34 + \text{etc.})^4 = (13 + 24 + \text{etc.})^2,$

or $= 25 + 36 + 47 = (23 + 34 + \text{etc.})^3.$

5). *ch.* $\phi\ (7\,;\ 4,\ 2,\ 1,\ 0)$;

$\phi = 35 + 46 + 57 = (34 + 45 + 56 + 67)^2,$

$= (12 + 34 + 45 + 56 + 67)^2,$

$$ch.\ \phi^{\frac{1}{2}} \begin{cases} (7\,;\ 3,\ 1,\ 1,\ 1,\ 1,\ 0) \\ (7\,;\ 2,\ 2,\ 1,\ 1,\ 1,\ 0). \end{cases}$$

6). *ch.* $\phi\ (7\,;\ 5,\ 2,\ 0)$;

$\phi = 16 + 27 = (12 + 23 + \text{etc.})^5,$

or $= 26 + 37 = (23 + 34 + \text{etc.})^4 = (24 + 35 + \text{etc.})^2,$

“ $= 36 + 47 = (34 + 45 + \text{etc.})^3 = (12 + 34 + 45 + \text{etc.})^3,$

“ $= 46 + 57 = (45 + 56 + 67)^2 = (12 + 45 + 56 + 67)^2.$

$$ch.\ \phi^{\frac{1}{2}} \begin{cases} (7\,;\ 4,\ 1,\ 1,\ 1,\ 0) \\ (7\,;\ 3,\ 2,\ 2,\ 0) \\ (7\,;\ 3,\ 2,\ 1,\ 1,\ 0), \end{cases} \qquad ch.\ \phi^{\frac{1}{3}} \begin{cases} (7\,;\ 3,\ 1,\ 1,\ 1,\ 1,\ 0) \\ (7\,;\ 2,\ 2,\ 1,\ 1,\ 1,\ 0). \end{cases}$$

7). *ch.* $\phi\ (7\,;\ 6,\ 1,\ 0)$;

$\phi = 17 = (12 + 23 + \text{etc.})^6 = (13 + 24 + \text{etc.})^3 = (14 + 25 + \text{etc.})^2,$

or $= 27 = (23 + 34 + \text{etc.})^5,$

“ $= 37 = (34 + 45 + 56 + 67)^4 = (12 + 34 + 45 + \text{etc.})^4$

$= (35 + 46 + \text{etc.})^2 = (12 + 35 + 46 + 57)^2,$

“ $= 47 = (45 + 56 + 67)^3 = (12 + 45 + 56 + 67)^3$

$= (12 + 23 + 45 + 56 + 67)^3,$

“ $= 57 = (56 + 67)^2.$

$$ch.\ \phi^{\frac{1}{2}} \begin{cases} (7\,;\ 5,\ 1,\ 1,\ 0) \\ (7\,;\ 4,\ 2,\ 1,\ 0) \\ (7\,;\ 3,\ 3,\ 1,\ 0), \end{cases} \quad ch.\ \phi^{\frac{1}{3}} \begin{cases} (7\,;\ 4,\ 1,\ 1,\ 1,\ 0) \\ (7\,;\ 3,\ 2,\ 1,\ 1,\ 0) \\ (7\,;\ 2,\ 2,\ 2,\ 1,\ 0), \end{cases} \quad ch.\ \phi^{\frac{1}{4}} \begin{cases} (7\,;\ 3,\ 1,\ 1,\ 1,\ 1,\ 0) \\ (7\,;\ 2,\ 2,\ 1,\ 1,\ 1,\ 0). \end{cases}$$

(f). *Matrix of order* 8.

1). *ch.* $\phi\ (8\,;\ 2,\ 2,\ 2,\ 2,\ 0)$;

$\phi = 13 + 24 + \text{etc.} = (12 + 23 + \text{etc.})^2.$

2). *ch.* $\phi\ (8\,;\ 3,\ 3,\ 2,\ 0)$;

$\phi = 14 + 25 + \text{etc.} = (12 + 23 + \text{etc.})^3$

3). *ch.* $\phi\ (8\,;\ 3,\ 2,\ 2,\ 1,\ 0)$;

$\phi = 24 + 35 + 46 + 57 + 68 = (23 + 34 + \text{etc.})^2$

4). *ch.* $\phi\ (8\,;\ 4,\ 4,\ 0)$;

$\phi = 15 + 26 + 37 + 48 = (12 + 23 + \text{etc.})^4 = (13 + 24 + \text{etc.})^2$

no. cu. root.

5). *ch.* ϕ $(8\,;\ 4,\ 3,\ 1,\ 0)$;

$\phi = 25 + 36 + 47 + 58 = (23 + 34 + 45 + \text{etc.})^3$,

or $= 13 + 46 + 57 + 68 = (12 + 23 + 45 + 56 + 67 + 78)^2$.

6). *ch.* ϕ $(8\,;\ 4,\ 2,\ 2,\ 0)$;

$\phi = 35 + 46 + 57 + 68 = (34 + 45 + 56 + \text{etc.})^2 = (12 + 34 + 45 + 56 + \text{etc.})^2$.

$$ch.\ \phi^{\frac{1}{2}} \begin{cases} (8\,;\ 3,\ 1,\ 1,\ 1,\ 1,\ 1,\ 0) \\ (8\,;\ 2,\ 2,\ 1,\ 1,\ 1,\ 1,\ 0). \end{cases}$$

7). *ch.* ϕ $(8\,;\ 5,\ 3,\ 0)$;

$\phi = 16 + 27 + 38 = (12 + 23 + \text{etc.})^5$,

or $= 26 + 37 + 48 = (23 + 34 + \text{etc.})^4 = (24 + 35 + \text{etc.})^2$,

" $= 36 + 47 + 58 = (34 + 45 + \text{etc.})^3 = (12 + 34 + 45 + \text{etc.})^3$.

$$ch.\ \phi^{\frac{1}{3}} \begin{cases} (8\,;\ 3,\ 1,\ 1,\ 1,\ 1,\ 1,\ 0) \\ (8\,;\ 2,\ 2,\ 1,\ 1,\ 1,\ 1,\ 0). \end{cases}$$

8). *ch.* ϕ $(8\,;\ 5,\ 2,\ 1,\ 0)$;

$\phi = 46 + 57 + 68 = (45 + 56 + 67 + 78)^2 = (12 + 45 + 56 + 67 + 78)^2$.

$$ch.\ \phi^{\frac{1}{2}} \begin{cases} (8\,;\ 4,\ 1,\ 1,\ 1,\ 1,\ 0) \\ (8\,;\ 3,\ 2,\ 1,\ 1,\ 1,\ 0). \end{cases}$$

9). *ch.* ϕ $(8\,;\ 6,\ 2,\ 0)$;

$\phi = 17 + 28 = (12 + 23 + 34 + \text{etc.})^6 = (13 + 24 + 35 + \text{etc.})^3$

$= (14 + 25 + \text{etc.})^2$,

or $= 27 + 38 = (23 + 34 + \text{etc.})^5$,

" $= 37 + 48 = (34 + 45 + 56 + \text{etc.})^4 = (12 + 34 + 45 + \text{etc.})^4$

$= (35 + 46 + \text{etc.})^2$,

" $= 47 + 58 = (45 + 56 + \text{etc.})^3 = (12 + 45 + 56 + \text{etc.})^3$

$= (12 + 23 + 45 + 56 + \text{etc.})^3$,

" $= 57 + 68 = (56 + 67 + 78)^2 = (12 + 56 + \text{etc.})^2$

$= (12 + 34 + 56 + \text{etc.})^2$.

$$ch.\ \phi^{\frac{1}{2}} \begin{cases} (8\,;\ 5,\ 1,\ 1,\ 1,\ 0) \\ (8\,;\ 4,\ 2,\ 2,\ 0) \\ (8\,;\ 4,\ 2,\ 1,\ 1,\ 0) \\ (8\,;\ 3,\ 3,\ 2,\ 0) \\ (8\,;\ 3,\ 3,\ 1,\ 1,\ 0), \end{cases} \qquad ch.\ \phi^{\frac{1}{3}} \begin{cases} (8\,;\ 4,\ 1,\ 1,\ 1,\ 1,\ 0) \\ (8\,;\ 3,\ 2,\ 1,\ 1,\ 1,\ 0) \\ (8\,;\ 2,\ 2,\ 2,\ 2,\ 0) \\ (8\,;\ 2,\ 2,\ 2,\ 1,\ 1,\ 0), \end{cases}$$

$$ch.\ \phi^{\frac{1}{4}} \begin{cases} (8\,;\ 3,\ 1,\ 1,\ 1,\ 1,\ 1,\ 0) \\ (8\,;\ 2,\ 2,\ 1,\ 1,\ 1,\ 1,\ 0). \end{cases}$$

10). *ch.* $\phi\ (8\,;\ 7,\ 1,\ 0)$;

$\phi = 18 = (12 + 23 + \text{etc.})^7$,

or $= 28 = (23 + 34 + \text{etc.})^6 = (24 + 35 + \text{etc.})^3 = (25 + 36 + \text{etc.})^2$,

" $= 38 = (34 + 45 + \text{etc.})^5 = (12 + 34 + 45 + \text{etc.})^5$,

" $= 48 = (45 + 56 + 67 + 78)^4 = (12 + 45 + \text{etc.})^4$
$= (12 + 23 + 45 + \text{etc.})^4 = (46 + 57 + 68)^2$
$= (12 + 46 + 57 + 68)^2$,

" $= 58 = (56 + 67 + 78)^3 = (12 + 56 + 67 + 78)^3$
$= (12 + 34 + 56 + 67 + 78)^3$,

" $= 68 = (67 + 78)^2 = (12 + 67 + 78)^2 = (12 + 34 + 67 + 78)^2$.

$$ch.\ \phi^{\frac{1}{2}} \left\{\begin{array}{l}(8\,;\ 6,\ 1,\ 1,\ 0)\\(8\,;\ 5,\ 2,\ 1,\ 0)\\(8\,;\ 4,\ 3,\ 1,\ 0),\end{array}\right. \qquad ch.\ \phi^{\frac{1}{3}} \left\{\begin{array}{l}(8\,;\ 5,\ 1,\ 1,\ 1,\ 0)\\(8\,;\ 4,\ 2,\ 1,\ 1,\ 0)\\(8\,;\ 3,\ 3,\ 1,\ 1,\ 0)\\(8\,;\ 3,\ 2,\ 2,\ 1,\ 0),\end{array}\right.$$

$$ch.\ \phi^{\frac{1}{4}} \left\{\begin{array}{l}(8\,;\ 4,\ 1,\ 1,\ 1,\ 1,\ 0)\\(8\,;\ 3,\ 2,\ 1,\ 1,\ 1,\ 0)\\(8\,;\ 2,\ 2,\ 2,\ 1,\ 1,\ 0),\end{array}\right. \qquad ch.\ \phi^{\frac{1}{5}} \left\{\begin{array}{l}(8\,;\ 3,\ 1,\ 1,\ 1,\ 1,\ 1,\ 0)\\(8\,;\ 2,\ 2,\ 1,\ 1,\ 1,\ 1,\ 0).\end{array}\right.$$

(g).—*Matrix of order* 9.

1). *ch.* $\phi\ (9\,;\ 2,\ 2,\ 2,\ 2,\ 1,\ 0)$;

$\phi = 13 + 24 + \text{etc.} = (12 + 23 + \text{etc.})^2$.

2). *ch.* $\phi\ (9\,;\ 3,\ 3,\ 3,\ 0)$;

$\phi = 14 + 25 + 36 + \text{etc.} = (12 + 23 + \text{etc.})^3$.

3). *ch.* $\phi\ (9\,;\ 3,\ 2,\ 2,\ 2,\ 0)$;

$\phi = 24 + 35 + 46 + \text{etc.} = (23 + 34 + \text{etc.})^2$.

4). *ch.* $\phi\ (9\,;\ 4,\ 4,\ 1,\ 0)$;

$\phi = 15 + 26 + 37 + 48 + 59 = (12 + 23 + \text{etc.})^4 = (13 + 24 + \text{etc.})^2$.

5). *ch.* $\phi\ (9\,;\ 4,\ 3,\ 2,\ 0)$;

$\phi = 25 + 36 + \text{etc.} = (23 + 34 + \text{etc.})^3$,

or $= 13 + 46 + 57 + 68 + 79 = (12 + 23 + 45 + 56 + \text{etc.})^2$.

6). *ch.* $\phi\ (9\,;\ 4,\ 2,\ 2,\ 1,\ 0)$;

$\phi = 35 + 46 + \text{etc.} = (34 + 45 + \text{etc.})^2 = (12 + 34 + 45 + \text{etc.})^2$

$$ch.\ \phi^{\frac{1}{2}} \left\{\begin{array}{l}(9\,;\ 3,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 0)\\(9\,;\ 2,\ 2,\ 1,\ 1,\ 1,\ 1,\ 1,\ 0).\end{array}\right.$$

7). *ch.* $\phi\,(9\,;\;5,\,4,\,0)$;

$$\phi = 16 + 27 + 38 + 49 = (12 + 23 + \text{etc.})^5,$$
$$\text{or } = 26+37+48+59 = (23+34+45+\text{etc.})^4 = (24+35+46+\text{etc.})^2.$$

8). *ch.* $\phi\,(9\,;\;5,\,3,\,1,\,0)$;

$$\phi = 36 + 47 + 58 + 69 = (34 + 45 + \text{etc.})^3 = (12 + 34 + \text{etc.})^3,$$
$$\text{or } = 13 + 57 + 68 + 79 = (12 + 23 + 57 + \text{etc.})^2.$$

$$ch.\ \phi^{\frac{1}{3}} \begin{cases} (9\,;\;3,\,1,\,1,\,1,\,1,\,1,\,1,\,0) \\ (9\,;\;2,\,2,\,1,\,1,\,1,\,1,\,1,\,0). \end{cases}$$

9). *ch.* $\phi\,(9\,;\;5,\,2,\,2,\,0)$;

$$\phi = 46 + 57 + 68 + 79 = (45 + 56 + \text{etc.})^2 = (12 + 45 + 56 + \text{etc.})^2.$$

$$ch.\ \phi^{\frac{1}{2}} \begin{cases} (9\,;\;4,\,1,\,1,\,1,\,1,\,1,\,0) \\ (9\,;\;3,\,2,\,1,\,1,\,1,\,1,\,0). \end{cases}$$

10). *ch.* $\phi\,(9\,;\;6,\,3,\,0)$;

$$\begin{aligned} \phi &= 17 + 28 + 39 = (12 + 23 + 34 + \text{etc.})^6 = (13 + 24 + \text{etc.})^3 \\ &\qquad = (14 + 25 + 36 + \text{etc.})^2, \\ \text{or } &= 27 + 38 + 49 = (23 + 34 + \text{etc.})^5, \\ \text{``} &= 37 + 48 + 59 = (34 + 45 + \text{etc.})^4 = (12 + 34 + 45 + \text{etc.})^4 \\ &\qquad = (35 + 46 + \text{etc.})^2 = (12 + 35 + 46 + \text{etc.})^2, \\ \text{``} &= 47 + 58 + 69 = (45 + 56 + \text{etc.})^3 = (12 + 45 + 56 + \text{etc.})^3 \\ &\qquad = (12 + 23 + 45 + 56 + \text{etc.})^3. \end{aligned}$$

$$ch.\ \phi^{\frac{1}{2}} \begin{cases} (9\,;\;4,\,2,\,2,\,1,\,0) \\ (9\,;\;3,\,3,\,3,\,0) \\ (9\,;\;3,\,3,\,2,\,1,\,0), \end{cases} \qquad ch.\ \phi^{\frac{1}{3}} \begin{cases} (9\,;\;4,\,1,\,1,\,1,\,1,\,1,\,0) \\ (9\,;\;3,\,2,\,1,\,1,\,1,\,1,\,0) \\ (9\,;\;2,\,2,\,2,\,1,\,1,\,1,\,0), \end{cases}$$

$$ch.\ \phi^{\frac{1}{4}} \begin{cases} (9\,;\;3,\,1,\,1,\,1,\,1,\,1,\,1,\,0) \\ (9\,;\;2,\,2,\,1,\,1,\,1,\,1,\,1,\,0). \end{cases}$$

11). *ch.* $\phi\,(9\,;\;6,\,2,\,1,\,0)$;

$$\begin{aligned} \phi &= 57 + 68 + 79 = (56 + 67 + \text{etc.})^2 = (12 + 56 + 67 + \text{etc.})^2 \\ &\qquad = (12 + 34 + 56 + 67 + \text{etc.})^2. \end{aligned}$$

$$ch.\ \phi^{\frac{1}{2}} \begin{cases} (9\,;\;5,\,1,\,1,\,1,\,1,\,0) \\ (9\,;\;4,\,2,\,1,\,1,\,1,\,0) \\ (9\,;\;3,\,3,\,1,\,1,\,1,\,0). \end{cases}$$

12). *ch.* $\phi\,(9;\ 7,\ 2,\ 0)$;

$$\begin{aligned}
\phi &= 18 + 29 = (12 + 23 + \text{etc.})^7, \\
\text{or } &= 28 + 39 = (23 + 34 + \text{etc.})^6 = (24 + 35 + \text{etc.})^3 \\
&\qquad\qquad = (25 + 36 + \text{etc.})^2, \\
\text{``} &= 38 + 49 = (34 + 45 + \text{etc.})^5 = (12 + 34 + 45 + \text{etc.})^5, \\
\text{``} &= 48 + 59 = (45 + 56 + \text{etc.})^4 = (12 + 45 + 56 + \text{etc.})^4 \\
&\qquad\qquad = (12 + 23 + 45 + 56 + \text{etc.})^4, \\
\text{``} &= 58 + 69 = (56 + 67 + \text{etc.})^3 = (12 + 56 + 67 + \text{etc.})^3 \\
&\qquad\qquad = (12 + 34 + 56 + \text{etc.})^3 = (12 + 23 + 56 + \text{etc.})^3, \\
\text{``} &= 67 + 79 = (67 + 78 + 89)^2 = (12 + 67 + 78 + 89)^2 \\
&\qquad\qquad = (12 + 34 + 67 + \text{etc.})^2, \\
\text{``} &= 13 + 79 = (12 + 23 + 78 + 89)^2.
\end{aligned}$$

$$ch.\ \phi^{\frac{1}{2}} \begin{cases} (9;\ 6,\ 1,\ 1,\ 1,\ 0) \\ (9;\ 5,\ 2,\ 2,\ 0) \\ (9;\ 5,\ 2,\ 1,\ 1,\ 0) \\ (9;\ 4,\ 3,\ 2,\ 0) \\ (9;\ 4,\ 3,\ 1,\ 1,\ 0), \end{cases} \qquad ch.\ \phi^{\frac{1}{3}} \begin{cases} (9;\ 5,\ 1,\ 1,\ 1,\ 1,\ 0) \\ (9;\ 4,\ 2,\ 1,\ 1,\ 1,\ 0) \\ (9;\ 3,\ 3,\ 1,\ 1,\ 1,\ 0) \\ (9;\ 3,\ 2,\ 2,\ 2,\ 0) \\ (9;\ 3,\ 2,\ 2,\ 1,\ 1,\ 0), \end{cases}$$

$$ch.\ \phi^{\frac{1}{4}} \begin{cases} (9;\ 4,\ 1,\ 1,\ 1,\ 1,\ 1,\ 0) \\ (9;\ 3,\ 2,\ 1,\ 1,\ 1,\ 1,\ 0) \\ (9;\ 2,\ 2,\ 2,\ 1,\ 1,\ 1,\ 0), \end{cases} \qquad ch.\ \phi^{\frac{1}{5}} \begin{cases} (9;\ 3,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 0) \\ (9;\ 2,\ 2,\ 1,\ 1,\ 1,\ 1,\ 1,\ 0). \end{cases}$$

13). *ch.* $\phi\,(9;\ 8,\ 1,\ 0)$;

$$\begin{aligned}
\phi &= 19 = (12 + 23 + \text{etc.})^8 = (13 + 24 + \text{etc.})^4 = (15 + 26 + \text{etc.})^2, \\
\text{or } &= 29 = (23 + 34 + \text{etc.})^7, \\
\text{``} &= 39 = (34 + 45 + 56 + \text{etc.})^6 = (12 + 34 + \text{etc.})^6 \\
&\qquad = (35 + 46 + \text{etc.})^3 = (12 + 35 + 46 + \text{etc.})^3 \\
&\qquad = (36 + 47 + \text{etc.})^2 = (12 + 36 + 47 + \text{etc.})^2, \\
\text{``} &= 49 = (45 + 56 + \text{etc.})^5 = (12 + 45 + 56 + \text{etc.})^5 \\
&\qquad = (12 + 23 + 45 + \text{etc.})^5, \\
\text{``} &= 59 = (56 + 67 + \text{etc.})^4 = (12 + 56 + 67 + \text{etc.})^4 \\
&\qquad = (12 + 23 + 56 + 67 + \text{etc.})^4 = (12 + 34 + 56 + 67 + \text{etc.})^4 \\
&\qquad = (12 + 23 + 34 + 56 + 67 + \text{etc.})^4 = (57 + 68 + \text{etc.})^2 \\
&\qquad = (12 + 57 + 68 + 79)^2 = (12 + 34 + 57 + \text{etc.})^2, \\
\text{``} &= 69 = (67 + 78 + 89)^3 = (12 + 67 + 78 + 89)^3 \\
&\qquad = (12 + 34 + 67 + \text{etc.})^3 = (12 + 23 + 67 + 78 + 89)^3 \\
&\qquad = (12 + 23 + 45 + \text{etc.})^3, \\
\text{``} &= 79 = (78 + 89)^2 = (12 + 78 + 89)^2.
\end{aligned}$$

$$ch.\ \phi^{\frac{1}{2}}\left\{\begin{array}{l}(9;\ 7,\ 1,\ 1,\ 0)\\(9;\ 6,\ 2,\ 1,\ 0)\\(9;\ 5,\ 3,\ 1,\ 0)\\(9;\ 4,\ 4,\ 1,\ 0),\end{array}\right.$$

$$ch.\ \phi^{\frac{1}{3}}\left\{\begin{array}{l}(9;\ 6,\ 1,\ 1,\ 1,\ 0)\\(9;\ 5,\ 2,\ 1,\ 1,\ 0)\\(9;\ 4,\ 3,\ 1,\ 1,\ 0)\\(9;\ 4,\ 2,\ 2,\ 1,\ 0)\\(9;\ 3,\ 3,\ 2,\ 1,\ 0),\end{array}\right.$$

$$ch.\ \phi^{\frac{1}{4}}\left\{\begin{array}{l}(9;\ 5,\ 1,\ 1,\ 1,\ 1,\ 0)\\(9;\ 4,\ 2,\ 1,\ 1,\ 1,\ 0)\\(9;\ 3,\ 3,\ 1,\ 1,\ 1,\ 0)\\(9;\ 2,\ 2,\ 2,\ 2,\ 1,\ 0),\end{array}\right.$$

$$ch.\ \phi^{\frac{1}{5}}\left\{\begin{array}{l}(9;\ 4,\ 1,\ 1,\ 1,\ 1,\ 1,\ 0)\\(9;\ 3,\ 2,\ 1,\ 1,\ 1,\ 1,\ 0)\\(9;\ 2,\ 2,\ 2,\ 1,\ 1,\ 1,\ 0),\end{array}\right.$$

$$ch.\ \phi^{\frac{1}{6}}\left\{\begin{array}{l}(9;\ 3,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 0)\\(9;\ 2,\ 2,\ 1,\ 1,\ 1,\ 1,\ 1,\ 0).\end{array}\right.$$

(h).—*Matrix of order* 10.

1). $ch.\ \phi\ (10;\ 2,\ 2,\ 2,\ 2,\ 2,\ 0)$;
$\phi = 13 + 24 + 35 + \text{etc.} = (12 + 23 + \text{etc.})^2$.

2). $ch.\ \phi\ (10;\ 3,\ 3,\ 3,\ 1,\ 0)$;
$\phi = 14 + 25 + \text{etc.} = (12 + 23 + \text{etc.})^3$.

3). $ch.\ \phi\ (10;\ 3,\ 2,\ 2,\ 2,\ 1,\ 0)$;
$\phi = 24 + 35 + \text{etc.} = (23 + 34 + \text{etc.})^2$.

4). $ch.\ \phi\ (10;\ 4,\ 4,\ 2,\ 0)$;
$\phi = 15 + 26 + \text{etc.} = (12 + 23 + \text{etc.})^4 = (13 + 24 + \text{etc.})^2$.

5). $ch.\ \phi\ (10;\ 4,\ 3,\ 3,\ 0)$;
$\phi = 25 + 36 + \text{etc.} = (23 + 34 + \text{etc.})^3$.

6). $ch.\ \phi\ (10;\ 4,\ 3,\ 2,\ 1,\ 0)$;
$\phi = 13 + 46 + 57 + \text{etc.} = (12 + 23 + \text{etc.})^2$.

7). $ch.\ \phi\ (10;\ 4,\ 2,\ 2,\ 2,\ 0)$;

$$\begin{aligned}\phi = 35 + 46 + 57 + \text{etc.}) &= (34 + 45 + \text{etc.})^2\\ &= (12 + 34 + 45 + \text{etc.})^2.\end{aligned}$$

$$ch.\ \phi^{\frac{1}{2}}\left\{\begin{array}{l}(10;\ 3,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 0)\\(10;\ 2,\ 2,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 0).\end{array}\right.$$

8). $ch.\ \phi\ (10;\ 5,\ 5,\ 0)$;
$\phi = 16 + 27 + \text{etc.} = (12 + 23 + \text{etc.})^5$.

9). $ch.\ \phi\ (10;\ 5,\ 4,\ 1,\ 0)$;
$\phi = 26 + 37 + \text{etc.} = (23 + 34 + \text{etc.})^4 = (24 + 35 + \text{etc.})^2$.

10). $ch.\ \phi\ (10;\ 5,\ 3,\ 2,\ 0)$;

$$\phi = 36 + 47 + \text{etc.} = (34 + 45 + \text{etc.})^3 = (12 + 34 + 45 + \text{etc.})^3,$$
$$\text{or } = 13 + 57 + \text{etc.} = (12 + 23 + 56 + 67 + \text{etc.})^2.$$

$$ch.\ \phi^{\frac{1}{3}} \begin{cases} (10;\ 3,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 0) \\ (10;\ 2,\ 2,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 0). \end{cases}$$

11). $ch.\ \phi\ (10;\ 5,\ 2,\ 2,\ 1,\ 0)$;

$$\phi = 46 + 57 + 68 + \text{etc.} = (45 + 56 + \text{etc.})^2 = (12 + 45 + 56 + \text{etc.})^2.$$

$$ch.\ \phi^{\frac{1}{2}} \begin{cases} (10;\ 4,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 0) \\ (10;\ 3,\ 2,\ 1,\ 1,\ 1,\ 1,\ 1,\ 0). \end{cases}$$

12). $ch.\ \phi\ (10;\ 6,\ 4,\ 0)$;

$$\phi = 17 + 28 + \text{etc.} = (12 + 23 + \text{etc.})^6 = (13 + 24 + \text{etc.})^3$$
$$= (14 + 25 + \text{etc.})^2,$$
$$\text{or } = 27 + 38 + \text{etc.} = (23 + 34 + \text{etc.})^5,$$
$$\text{“} = 37 + 48 + \text{etc.} = (34 + 45 + \text{etc.})^4 = (12 + 34 + 45 + \text{etc.})^4$$
$$= (35 + 46 + \text{etc.})^2 = (12 + 35 + 46 + \text{etc.})^2,$$
$$\text{“} = 14 + 58 + 69 + 710 = (12 + 23 + 34 + 56 + 67 + \text{etc.})^3.$$

$$ch.\ \phi^{\frac{1}{2}} \begin{cases} (10;\ 4,\ 2,\ 2,\ 2,\ 0) \\ (10;\ 3,\ 3,\ 3,\ 1,\ 0) \\ (10;\ 3,\ 3,\ 2,\ 2,\ 0), \end{cases} \qquad ch.\ \phi^{\frac{1}{3}} \begin{cases} (10;\ 2,\ 2,\ 2,\ 2,\ 2,\ 0) \\ (10;\ 2,\ 2,\ 2,\ 2,\ 1,\ 1,\ 0). \end{cases}$$

$$ch.\ \phi^{\frac{1}{4}} \begin{cases} (10;\ 3,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 0) \\ (10;\ 2,\ 2,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 0). \end{cases}$$

13). $ch.\ \phi\ (10;\ 6,\ 3,\ 1,\ 0)$;

$$\phi = 47 + 58 + \text{etc.} = (45 + 56 + \text{etc.})^3 = (12 + 45 + \text{etc.})^3$$
$$= (12 + 23 + 45 + \text{etc.})^3,$$
$$\text{or } = 13 + 68 + 79 + 810 = (12 + 23 + 67 + 78 + \text{etc.})^2$$
$$= (12 + 23 + 45 + 67 + 78 + \text{etc.})^2.$$

$$ch.\ \phi^{\frac{1}{2}} \begin{cases} (10;\ 4,\ 2,\ 2,\ 1,\ 1,\ 0) \\ (10;\ 3,\ 3,\ 2,\ 1,\ 1,\ 0), \end{cases} \qquad ch.\ \phi^{\frac{1}{3}} \begin{cases} (10;\ 4,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 0) \\ (10;\ 3,\ 2,\ 1,\ 1,\ 1,\ 1,\ 1,\ 0) \\ (10;\ 2,\ 2,\ 2,\ 1,\ 1,\ 1,\ 1,\ 0). \end{cases}$$

14). $ch.\ \phi\ (10;\ 6,\ 2,\ 2,\ 0)$;

$$\phi = 57 + 68 + 79 + 810 = (56 + 67 + \text{etc.})^2 = (12 + 56 + \text{etc.})^2$$
$$= (12 + 34 + 56 + \text{etc.})^2$$

$$ch.\ \phi^{\frac{1}{2}}\begin{cases}(10;\ 5,\ 1,\ 1,\ 1,\ 1,\ 1,\ 0)\\(10;\ 4,\ 2,\ 1,\ 1,\ 1,\ 1,\ 0)\\(10;\ 3,\ 3,\ 1,\ 1,\ 1,\ 1,\ 0).\end{cases}$$

15). *ch.* $\phi\,(10;\ 7,\ 3,\ 0)$;

$$\begin{aligned}\phi &= 18+29+310=(12+23+\text{etc.})^7,\\ \text{or } &= 28+39+410=(23+34+\text{etc.})^6=(24+35+\text{etc.})^3\\ &\qquad\qquad\qquad\;\, =(25+36+\text{etc.})^2,\\ \text{“} &= 38+49+510=(34+45+\text{etc.})^5=(12+34+45+\text{etc.})^5,\\ \text{“} &= 48+59+610=(45+56+\text{etc.})^4=(12+45+56+\text{etc.})^4\\ &\qquad\qquad\qquad\;\, =(12+23+45+\text{etc.})^4=(46+57+\text{etc.})^2\\ &\qquad\qquad\qquad\;\, =(12+46+57+\text{etc.})^2,\\ \text{“} &= 58+69+710=(56+67+\text{etc.})^3=(12+56+67+\text{etc.})^3\\ &\qquad\qquad\qquad\;\, =(12+23+56+\text{etc.})^3=(12+34+56+\text{etc.})^3.\end{aligned}$$

$$ch.\ \phi^{\frac{1}{2}}\begin{cases}(10;\ 5,\ 2,\ 2,\ 1,\ 0)\\(10;\ 4,\ 3,\ 3,\ 0)\\(10;\ 4,\ 3,\ 2,\ 1,\ 0),\end{cases}\qquad ch.\ \phi^{\frac{1}{3}}\begin{cases}(10;\ 5,\ 1,\ 1,\ 1,\ 1,\ 1,\ 0)\\(10;\ 4,\ 2,\ 1,\ 1,\ 1,\ 1,\ 0)\\(10;\ 3,\ 3,\ 1,\ 1,\ 1,\ 1,\ 0)\\(10;\ 3,\ 2,\ 2,\ 2,\ 1,\ 0)\\(10;\ 3,\ 2,\ 2,\ 1,\ 1,\ 1,\ 0).\end{cases}$$

$$ch.\ \phi^{\frac{1}{4}}\begin{cases}(10;\ 4,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 0)\\(10;\ 3,\ 2,\ 1,\ 1,\ 1,\ 1,\ 1,\ 0)\\(10;\ 2,\ 2,\ 2,\ 1,\ 1,\ 1,\ 1,\ 0),\end{cases}\qquad ch.\ \phi^{\frac{1}{5}}\begin{cases}(10;\ 3,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 0)\\(10;\ 2,\ 2,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 0).\end{cases}$$

16). *ch.* $\phi\,(10;\ 7,\ 2,\ 1,\ 0)$;

$$\begin{aligned}\phi &= 68+79+810=(67+78+\text{etc.})^2=(12+67+78+\text{etc.})^2\\ &\qquad\qquad\qquad\;\, =(12+34+67+78+\text{etc.})^2.\end{aligned}$$

$$ch.\ \phi^{\frac{1}{2}}\begin{cases}(10;\ 6,\ 1,\ 1,\ 1,\ 1,\ 0)\\(10;\ 5,\ 2,\ 1,\ 1,\ 1,\ 0)\\(10;\ 4,\ 3,\ 1,\ 1,\ 1,\ 0).\end{cases}$$

17). *ch.* $\phi\,(10;\ 8,\ 2,\ 0)$;

$$\begin{aligned}\phi &= 19+210=(12+23+\text{etc.})^8=(13+24+\text{etc.})^4\\ &\qquad\qquad\;\, =(15+26+\text{etc.})^2,\\ \text{or } &= 29+310=(23+34+\text{etc.})^7,\\ \text{“} &= 39+410=(34+45+\text{etc.})^6=(12+34+45+\text{etc.})^6\\ &\qquad\qquad\;\, =(35+46+\text{etc.})^3=(12+35+\text{etc.})^3\\ &\qquad\qquad\;\, =(36+47+\text{etc.})^2=(12+36+47+\text{etc.})^2,\end{aligned}$$

$$\begin{aligned}
\text{or } &= 49 + 510 = (45 + 56 + 67 + \text{etc.})^5 = (12 + 45 + 56 + \text{etc.})^5 \\
& = (12 + 23 + 45 + \text{etc.})^5, \\
\text{``} &= 59 + 610 = (56 + 67 + \text{etc.})^4 = (12 + 56 + 67 + \text{etc.})^4 \\
& = (12 + 23 + 56 + \text{etc.})^4 = (12 + 34 + 56 + \text{etc.})^4 \\
& = (12 + 23 + 34 + 56 + \text{etc.})^4 = (57 + 68 + \text{etc.})^2 \\
& = (12 + 57 + 68 + \text{etc.})^2 = (12 + 34 + 57 + \text{etc.})^2, \\
\text{``} &= 69 + 710 = (67 + 78 + \text{etc.})^3 = (12 + 67 + 78 + \text{etc.})^3 \\
& = (12 + 23 + 67 + 78 + \text{etc.})^3 \\
& = (12 + 23 + 45 + 67 + 78 + \text{etc.})^3 \\
& = (12 + 34 + 67 + 78 + \text{etc.})^3, \\
\text{``} &= 79 + 810 = (78 + 89 + 910)^2 = (12 + 78 + 89 + 910)^2 \\
& = (12 + 34 + 78 + \text{etc.})^2 = (12+34+56+78+ \text{etc.})^2.
\end{aligned}$$

$$ch.\ \phi^{\frac{1}{2}} \begin{cases} (10;\ 7,\ 1,\ 1,\ 1,\ 0) \\ (10;\ 6,\ 2,\ 1,\ 1,\ 0) \\ (10;\ 6,\ 2,\ 2,\ 0) \\ (10;\ 5,\ 3,\ 2,\ 0) \\ (10;\ 5,\ 3,\ 1,\ 1,\ 0) \\ (10;\ 4,\ 4,\ 2,\ 0) \\ (10;\ 4,\ 4,\ 1,\ 1,\ 0), \end{cases} \qquad ch.\ \phi^{\frac{1}{3}} \begin{cases} (10;\ 6,\ 1,\ 1,\ 1,\ 1,\ 0) \\ (10;\ 5,\ 2,\ 1,\ 1,\ 1,\ 0) \\ (10;\ 4,\ 3,\ 1,\ 1,\ 1,\ 0) \\ (10;\ 4,\ 2,\ 2,\ 2,\ 0) \\ (10;\ 4,\ 2,\ 2,\ 1,\ 1,\ 0) \\ (10;\ 3,\ 3,\ 2,\ 2,\ 0) \\ (10;\ 3,\ 3,\ 2,\ 1,\ 1,\ 0), \end{cases}$$

$$ch.\ \phi^{\frac{1}{4}} \begin{cases} (10;\ 5,\ 1,\ 1,\ 1,\ 1,\ 1,\ 0) \\ (10;\ 4,\ 2,\ 1,\ 1,\ 1,\ 1,\ 0) \\ (10;\ 3,\ 3,\ 1,\ 1,\ 1,\ 1,\ 0) \\ (10;\ 3,\ 2,\ 2,\ 1,\ 1,\ 1,\ 0) \\ (10;\ 2,\ 2,\ 2,\ 2,\ 2,\ 0) \\ (10;\ 2,\ 2,\ 2,\ 2,\ 1,\ 1,\ 0), \end{cases} \qquad \begin{aligned} &ch.\ \phi^{\frac{1}{5}} \begin{cases} (10;\ 4,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 0) \\ (10;\ 3,\ 2,\ 1,\ 1,\ 1,\ 1,\ 1,\ 0) \\ (10;\ 2,\ 2,\ 2,\ 1,\ 1,\ 1,\ 1,\ 0), \end{cases} \\ &ch.\ \phi^{\frac{1}{6}} \begin{cases} (10;\ 3,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 0) \\ (10;\ 2,\ 2,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 0). \end{cases} \end{aligned}$$

18). $ch.\ \phi\ (10;\ 9,\ 1,\ 0)$;

$$\begin{aligned}
\phi &= 110 = (12 + 23 + \text{etc.})^9 = (14 + 25 + \text{etc.})^3, \\
\text{or } &= 210 = (23 + 34 + \text{etc.})^8 = (24 + 35 + \text{etc.})^4 \\
& = (26 + 37 + \text{etc.})^2, \\
\text{``} &= 310 = (34 + 45 + \text{etc.})^7 = (12 + 34 + 45 + \text{etc.})^7, \\
\text{``} &= 410 = (45 + 56 + \text{etc.})^6 = (12 + 45 + \text{etc.})^6 \\
& = (12 + 23 + 45 + 56 + \text{etc.})^6 = (46 + 57 + \text{etc.})^3 \\
& = (12 + 46 + 57 + \text{etc.})^3 = (12 + 23 + 46 + \text{etc.})^3 \\
& = (47 + 58 + \text{etc.})^2 = (12 + 47 + \text{etc.})^2 = \text{etc.},
\end{aligned}$$

$$\begin{aligned}
\text{or} = 510 &= (56 + 67 + \text{etc.})^5 = (12 + 56 + \text{etc.})^5\\
&= (12 + 23 + 56 + \text{etc.})^5 = (12 + 34 + 56 + \text{etc.})^5\\
&= (12 + 23 + 34 + 56 + \text{etc.})^5,\\
\text{``} = 610 &= (67 + 78 + \text{etc.})^4 = (12 + 67 + 78 + \text{etc.})^4\\
&= (12 + 34 + 67 + \text{etc.})^4 = (12 + 23 + 67 + 78 + \text{etc.})^4\\
&= (12 + 23 + 34 + 67 + 78 + \text{etc.})^4\\
&= (12 + 23 + 45 + 67 + \text{etc.})^4 = (68 + 79 + 810)^2\\
&= (12 + 68 + 79 + \text{etc.})^2 = (12 + 34 + 68 + \text{etc.})^2,\\
\text{``} = 710 &= (78 + 89 + 910)^3 = (12 + 78 + \text{etc.})^3\\
&= (12 + 23 + 78 + \text{etc.})^3 = (12 + 34 + 56 + 78 + \text{etc.})^3\\
&= (12 + 23 + 45 + 78\ \text{etc.})^3\\
&= (12 + 23 + 45 + 56 + 78 + \text{etc.})^3\\
&= (12 + 34 + 56 + 78 + \text{etc.})^3,\\
\text{``} = 810 &= (89 + 910)^2 = (12 + 89 + 910)^2\\
&= (12 + 34 + 89 + 910)^2 = (12 + 34 + 56 + 89 + 910)^2.
\end{aligned}$$

$$ch.\ \phi^{\frac{1}{2}} \begin{cases} (10;\ 8,\ 1,\ 1,\ 0)\\ (10;\ 7,\ 2,\ 1,\ 0)\\ (10;\ 6,\ 3,\ 1,\ 0)\\ (10;\ 5,\ 4,\ 1,\ 0), \end{cases}
\qquad
ch.\ \phi^{\frac{1}{3}} \begin{cases} (10;\ 7,\ 1,\ 1,\ 1,\ 0)\\ (10;\ 6,\ 2,\ 1,\ 1,\ 0)\\ (10;\ 5,\ 3,\ 1,\ 1,\ 0)\\ (10;\ 5,\ 2,\ 2,\ 1,\ 0)\\ (10;\ 4,\ 4,\ 1,\ 1,\ 0)\\ (10;\ 4,\ 3,\ 2,\ 1,\ 0)\\ (10;\ 3,\ 3,\ 3,\ 1,\ 0), \end{cases}$$

$$ch.\ \phi^{\frac{1}{4}} \begin{cases} (10;\ 6,\ 1,\ 1,\ 1,\ 1,\ 0)\\ (10;\ 5,\ 2,\ 1,\ 1,\ 1,\ 0)\\ (10;\ 4,\ 3,\ 1,\ 1,\ 1,\ 0)\\ (10;\ 4,\ 2,\ 2,\ 1,\ 1,\ 0)\\ (10;\ 3,\ 3,\ 2,\ 1,\ 1,\ 0)\\ (10;\ 3,\ 2,\ 2,\ 2,\ 1,\ 0), \end{cases}
\qquad
ch.\ \phi^{\frac{1}{5}} \begin{cases} (10;\ 5,\ 1,\ 1,\ 1,\ 1,\ 1,\ 0)\\ (10;\ 4,\ 2,\ 1,\ 1,\ 1,\ 1,\ 0)\\ (10;\ 3,\ 3,\ 1,\ 1,\ 1,\ 1,\ 0)\\ (10;\ 2,\ 2,\ 2,\ 2,\ 1,\ 1,\ 0) \end{cases}$$

$$ch.\ \phi^{\frac{1}{6}} \begin{cases} (10;\ 4,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 0)\\ (10;\ 3,\ 2,\ 1,\ 1,\ 1,\ 1,\ 1,\ 0)\\ (10;\ 2,\ 2,\ 2,\ 1,\ 1,\ 1,\ 1,\ 0), \end{cases}
\qquad
ch.\ \phi^{\frac{1}{7}} \begin{cases} (10;\ 3,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 0)\\ (10;\ 2,\ 2,\ 1,\ 1,\ 1,\ 1,\ 1,\ 0). \end{cases}$$

29. Proceeding now to the consideration of matrices having some latent roots zero and others different from zero it may be observed that:

(1). The nullity of N depends on the nullity of ϕ and in the following way:

$$N_y[N^\nu] = N_y[\phi^\nu] + \omega - p_s.$$

And consequently when the nullity of ϕ is equal to its vacuity the nullity of N is ω, and therefore N vanishes.

(2). When the nullity of ϕ is equal to its vacuity it has an n^{th} root, and when the nullity is less than the vacuity ϕ obviously cannot have a root with index greater than its nullity.

(3). If $N[\phi] = p$, ϕ cannot have a $(p-\mu)^{\text{th}}$ root unless N has a $(p-\mu)^{\text{th}}$ root.

30. N is a nilpotent matrix — a root of zero — such as was considered in Art. 28, and as was there shown will have a q^{th} root unless the law of nullity prohibits. The relation existing between the nullities of N and ϕ shows us that if the law of nullity permits one it will also permit the other to have a q^{th} root, and consequently we have the theorem that: There will always be a q^{th} root unless the law of nullity prohibits.

§6.—*Transcendental Functions of a Matrix.*

31. In this section I shall consider a few cases of the elementary transcendental functions of a matrix.

(a). *Exponential function.*—I define e^ϕ by the ordinary series, viz.:

$$\begin{aligned} e^\phi &= 1 + \frac{\phi}{1} + \frac{\phi^2}{2!} + \frac{\phi^3}{3!} + \text{etc.} + \frac{\phi^n}{n!} + \text{etc.} \\ &= \sum_0^\infty {}_\mu \frac{\phi^\mu}{\mu!}. \end{aligned}$$

Let $\phi^n = (A_0 + nA_1 + n^2A_2 + \ldots\ldots + n^{p_1}A_{p_1})g_1^n + (B_0 + nB_1 + \ldots\ldots + n^{p_2}B_{p_2})g_2^n$
$\qquad + \ldots\ldots + (S_0 + nS_1 + n^2S_2 + \ldots\ldots + n^{p_s}S_{p_s})g_s^n$,

then

$$e^\phi = \sum_0^\infty {}_\mu \left\{ \frac{(A_0 + \mu A_1 + \ldots\ldots + \mu^{p_1}A_{p_1})g_1^\mu + \ldots\ldots + (S_0 + \mu S_1 + \ldots\ldots + \mu^{p_s}S_{p_s})g_s^\mu}{\mu!} \right\}$$

$$
\begin{aligned}
&= \sum_{0}^{\infty}{}^{\mu} A_0 \frac{g_1^\mu}{\mu\,!} + \sum_{0}^{\infty}{}^{\mu} \mu A_1 \frac{g_1^\mu}{\mu\,!} + \sum_{0}^{\infty}{}^{\mu} \mu^2 A_2 \frac{g_1^\mu}{\mu\,!} + \ldots\ldots + \sum_{0}^{\infty}{}^{\mu} \mu^{p_1} A_{p_1} \frac{g_1^\mu}{\mu\,!} \\
&\quad + \sum_{0}^{\infty}{}^{\mu} B_0 \frac{g_2^\mu}{\mu\,!} + \sum_{0}^{\infty}{}^{\mu} \mu B_1 \frac{g_2^\mu}{\mu\,!} + \ldots\ldots + \sum_{0}^{\infty}{}^{\mu} \mu^{p_2} B_{p_2} \frac{g_2^\mu}{\mu\,!} \\
&\quad + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
&\quad + \sum_{0}^{\infty}{}^{\mu} S_0 \frac{g_s^\mu}{\mu\,!} + \sum_{0}^{\infty}{}^{\mu} \mu S_1 \frac{g_s^\mu}{\mu\,!} + \ldots\ldots + \sum_{0}^{\infty}{}^{\mu} \mu^{p_s} S_{p_s} \frac{g_s^\mu}{\mu\,!} \\
&= A_0 e^{g_1} + g_1 A_1 e^{g_1} + \ldots\ldots + \sum_{1}^{p_1}{}^{\lambda} g_1^\lambda A_{p_1} e^{g_1} \\
&\quad + B_0 e^{g_2} + g_2 B_1 e^{g_2} + \ldots\ldots + \sum_{1}^{p_2}{}^{\lambda} g_2^\lambda B_{p_2} e^{g_2} \\
&\quad + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
&\quad + S_0 e^{g_s} + g_s S_1 e^{g_s} + \ldots\ldots + \sum_{1}^{p_s}{}^{\lambda} g_s^\lambda S_{p_s} e^{g_s}.
\end{aligned}
$$

Again

$$
\begin{aligned}
e^{-\phi} &= \sum_{0}^{\infty}{}^{\mu} (-1)^{\mu+2} \frac{\Phi^\mu}{\mu\,!} \\
&= \sum_{0}^{\infty}{}^{\mu} (-1)^{\mu+2} \left\{ \frac{(A_0 + \mu A_1 + \ldots + \mu^{p_1} A_{p_1}) g_1^\mu + \ldots + (S_0 + \mu S_1 + \ldots + \mu^{p_s} S_{p_s}) g_s^\mu}{\mu\,!} \right\} \\
&= A_0 \sum_{0}^{\infty}{}^{\mu} (-1)^{\mu+2} \frac{g_1^\mu}{\mu\,!} + \ldots\ldots + \sum_{0}^{\infty}{}^{\mu} (-1)^{\mu+2} \mu^{p_1} A_{p_1} \frac{g_1^\mu}{\mu\,!} \\
&\quad + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
&\quad + S_0 \sum_{0}^{\infty}{}^{\mu} (-1)^{\mu+2} \frac{g_s^\mu}{\mu\,!} + \ldots\ldots + \sum_{0}^{\infty}{}^{\mu} (-1)^{\mu+2} \mu^{p_s} S_{p_s} \frac{g_s^\mu}{\mu\,!}, \\
&= \{ A_0 - g_1 A_1 - (g_1 - g_1^2) A_2 - \ldots\ldots - \sum_{1}^{p_1}{}^{\mu} (-1)^{\mu+1} g_1^\mu A_{p_1} \} e^{-g_1} \\
&\quad + \{ B_0 - g_2 B_1 - (g_2 - g_2^2) B_2 - \ldots\ldots - \sum_{1}^{p_2}{}^{\mu} (-1)^{\mu+1} g_2^\mu B_{p_2} \} e^{-g_2} \\
&\quad + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
&\quad + \{ S_0 - g_s S_1 - (g_s - g_s^2) S_2 - \ldots\ldots - \sum_{1}^{p_s}{}^{\mu} (-1)^{\mu+1} g_s^\mu S_{p_s} \} e^{-g_s}.
\end{aligned}
$$

(b). *Logarithmic function.*—I define $\log\phi$ by the series:

$$\log\phi = (\phi - 1) - \tfrac{1}{2}(\phi - 1)^2 + \tfrac{1}{3}(\phi - 1)^3 - \tfrac{1}{4}(\phi - 1)^4 + \text{etc.}$$

$$= \sum_{1}^{\infty}{}_{\lambda} (-1)^{\lambda+1}\frac{(\phi - 1)^{\lambda}}{\lambda}.$$

Let $\quad \phi^n = A_0 g^n + B_0 g_2^n + \ldots . + W_0 g_{\omega}^n,$

then $\quad (\phi - 1)^{\lambda} = A_0(g_1 - 1)^{\lambda} + B_0(g_2 - 1)^{\lambda} + \ldots . + W_0(g_{\omega} - 1)^{\lambda}$

and

$$\log\phi = \sum_{1}^{\infty}{}_{\lambda} (-1)^{\lambda+1}\left\{\frac{A_0(g_1 - 1)^{\lambda} + B_0(g_2 - 1)^{\lambda} + \ldots . + W_0(g_{\omega} - 1)^{\lambda}}{\lambda}\right\}$$

$$= A_0\sum_{1}^{\infty}{}_{\lambda}\frac{(g_1 - 1)^{\lambda}}{\lambda} + \ldots . + W_0\sum_{1}^{\infty}{}_{\lambda}\frac{(g_s - 1)^{\lambda}}{\lambda}$$

$$= A_0 \log g_1 + B_0 \log g_2 + \ldots . + W_0 \log g_{\omega}.$$

Again let $\quad \phi^n = (A_0 + nA_1)g_1^n + (B_0 + nB_1)g_2^n + \ldots . + (S_0 + nS_1)g_s^n$

then $\quad (\phi - 1)^{\lambda} = A_0(g_1 - 1)^{\lambda} + \lambda A_1 g_1(g_1 - 1)^{\lambda-1} + B_0(g_2 - 1)^{\lambda} + \lambda B_1 g_2(g_2 - 1)^{\lambda-1}$
$\quad + \ldots . + S_0(g_s - 1)^{\lambda} + \lambda S_1 g_2(g_2 - 1)^{\lambda-1},$

and $\quad \log\phi = A_0 \log g_1 + B_0 \log g_2 + \ldots . + S_0 \log g_s$
$\quad + A_1 \quad + B_1 \qquad + \ldots . + S_1.$

Having defined e^{ϕ} and $\log\phi$ independently, let us see what $e^{\log\phi}$ becomes, using the definition given for $\log\phi$.

We have

$$\begin{aligned} e^{\log\phi} &= e^{(A_0 \log g_1 + B_0 \log g_2 + \ldots . + W_0 \log g_{\omega})} \\ &= A_0 e^{\log g_1} + B_0 e^{\log g_2} + \ldots . + W_0 e^{\log g_{\omega}} \\ &= A_0 g_1 + B_0 g_2 \qquad + \ldots . + W_0 g_{\omega} \\ &= \phi, \end{aligned}$$

which shows that having defined e^{ϕ} by the ordinary series, we were justified in defining $\log\phi$ as in (b).

(c). *Sin* ϕ.—I define $\sin\phi$ as follows:

$$\sin\phi = \frac{\phi}{1} - \frac{\phi^3}{3!} + \frac{\phi^5}{5!} - \frac{\phi^7}{7!} + \text{etc.}$$

$$= \sum_{0}^{\infty}{}_{\lambda} (-1)^{\lambda}\frac{\phi^{2\lambda+1}}{(2\lambda + 1)!}.$$

Let $\phi^n = A_0 g_1^n + B_0 g_2^n + \ldots\ldots + W_0 g_\omega^n$,

then
$$\sin\phi = \sum_0^\infty{}^\lambda (-1)^\lambda \left\{ \frac{A_0 g_1^{2\lambda+1} + B_0 g_2^{2\lambda+1} + \ldots\ldots + W_0 g_\omega^{2\lambda+1}}{(2\lambda+1)!} \right\}$$
$$= A_0 \sin g_1 + B_0 \sin g_2 + \ldots\ldots + W_0 \sin g_\omega .$$

Again if $\phi^n = (A_0 + nA_1) g_1^n + (B_0 + nB_1) g_2^n + \ldots\ldots + (S_0 + nS_1) g_s^n$,

then
$$\sin\phi = A_0 \sin g_1 + B_0 \sin g_2 + \ldots\ldots + S_0 \sin g_s$$
$$+ g_1 A_1 \cos g_1 + g_2 B_1 \cos g_2 + \ldots\ldots + g_s S_1 \cos g_s .$$

(d). *Sin*$^{-1}\phi$.—I define $\sin^{-1}\phi$ as follows:

$$\sin^{-1}\phi = \frac{\phi}{1} + \frac{1}{2}\cdot\frac{\phi^3}{3} + \frac{1.3}{2.4}\cdot\frac{\phi^5}{5} + \ldots\ldots + \frac{1.3\ldots\ldots\overline{2r-1}}{2.4\ldots\ldots 2r}\cdot\frac{x^{2r+1}}{2r+1} + \ldots\ldots$$
$$= \sum_0^\infty{}^\lambda \frac{1.3\ldots\ldots\overline{2\lambda-1}}{2.4\ldots\ldots 2\lambda}\cdot\frac{\phi^{2\lambda+1}}{2\lambda+1} .$$

Let $\phi^n = A_0 g_1^n + B_0 g_2^n + \ldots\ldots + W_0 g_\omega^n$,

then
$$\sin^{-1}\phi = \sum_0^\infty{}^\lambda \frac{1.3\ldots\ldots\overline{2\lambda-1}}{2.4\ldots\ldots 2\lambda}\cdot\left\{ \frac{A_0 g_1^{2\lambda+1} + B_0 g_2^{2\lambda+1} + \ldots\ldots + W_0 g_\omega^{2\lambda+1}}{2\lambda+1} \right\}.$$
$$= A_0 \sin^{-1} g_1 + B_0 \sin^{-1} g_2 + \ldots\ldots + W_0 \sin^{-1} g_\omega .$$

Having defined $\sin\phi$ and $\sin^{-1}\phi$ independently, let us see what $\sin\sin^{-1}\phi$ becomes, using the definition given for $\sin^{-1}\phi$.

We have

$$\sin\sin^{-1}\phi = \sin\sum_0^\infty{}^\lambda \frac{1.3\ldots\ldots\overline{2\lambda-1}}{2.4\ldots\ldots 2\lambda}\cdot\frac{\phi^{2\lambda+1}}{2\lambda+1}$$
$$= \sin\sum_1^\infty{}^\lambda \frac{1.3\ldots\ldots\overline{2\lambda-1}}{2.4\ldots\ldots 2\lambda}\cdot\left\{ \frac{A_0 g_1^{2\lambda+1} + B_0 g_2^{2\lambda+1} + \ldots\ldots + W_0 g_\omega^{2\lambda+1}}{2\lambda+1} \right\}$$
$$= \sin\left\{ A_0 \sin^{-1} g_1 + B_0 \sin^{-1} g_2 + \ldots\ldots + W_0 \sin^{-1} g_\omega \right\}$$
$$= A_0 g_1 + B_0 g_2 + \ldots\ldots + W_0 g_\omega$$
$$= \phi,$$

which shows that having defined $\sin\phi$ by the ordinary series we were justified in defining $\sin^{-1}\phi$ as in (d).

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